Regional Mathematical Olympiad Solutions – 2019
(20-10-2019)

1. Suppose x is a nonzero real number such that both x^5 and $20x+\frac{19}{x}$ are rational numbers. Prove that x is a rational number.

Sol. $x^5, 20x+\frac{19}{x} \in Q$

$20x^5, \frac{19}{x^5} \in Q \Rightarrow 20x^5+\frac{19}{x^5} \in Q$

Also $\frac{19}{x} \in Q \Rightarrow \left(\frac{19}{x}\right)^5 \in Q$

$\frac{19^5}{x^5} \in Q \Rightarrow \frac{1}{x^5} \in Q$

Let p(t) = $t^5 - \left(\frac{19}{x}\right)^5$ and

q(t) = $\left(20x+\frac{19}{x}-t\right)^5 - (20x)^5$

Note that $\frac{19}{x}$ is a common root of both poly.

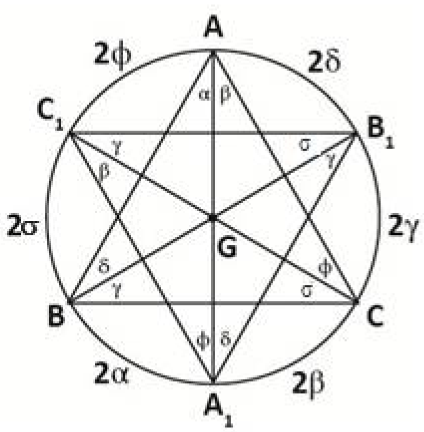
Also p(t), q(t) $\in$ Z[t]

Hence their common factor $\left(t-\frac{19}{x}\right) \in Z[t]$

$\Rightarrow \frac{19}{x} \in Z \qquad \Rightarrow$ x t Z

2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$. $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

Sol.



Given $\angle BAC = \angle A_1B_1C_1$

$\angle ABC = \angle A_1C_1B_1$

$\angle ACB = \angle B_1A_1C_1$

PT ΔADC, $\Delta A_1B_1C_1$ are equilateral Δ

Let arc's BA_1, A_1C, CB_1, B_1A, AC_1, C_1B be of mean 2α, 2β, 2γ, 2δ, 2ϕ, and 26 resp.

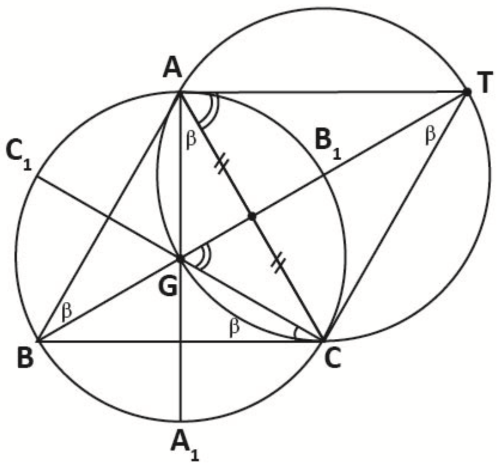
$\alpha + \beta = \angle BAC = \angle A_1B_1C_1 = \sigma + \alpha \Rightarrow \beta = \sigma$

$\gamma + \delta = \angle ABC = \angle A_1C_1B_1 = \beta + \gamma \Rightarrow \beta = \delta$

$\phi + \delta = \angle ACB = \angle B_1A_1C_1 = \phi + \delta \Rightarrow \sigma = \delta$

$\Rightarrow \beta = \sigma = \delta$

Now TPT $\alpha = \gamma = \phi$



Let $\odot$ AGC meet BB, at T$\angle GTC = \angle GAC = \beta = \angle ABT = \delta$

$\therefore$ AB ||TC, Also $M = M_{AC}$

So $\Delta ABM \cong \Delta CTM$

$\therefore \square$ ABCT is a $||^m$ so AT || C

$\angle TAC = \angle TGC = \gamma + \delta$

$= \angle ACB = \delta + \phi$

$\Rightarrow \gamma = \phi$ as $\sigma = \delta$.

Similarly we can prove $\alpha = \gamma = \phi$

$\therefore$ All 3 angles of Δ's are equal.

3. Let a, b, c be positive real numbers such that a + b + c = 1. Prove that

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3}\le\frac{1}{5abc}$$

Sol. Since a + b + c = 1

$$\sum \frac{a}{a^2+b^3+c^3} = \sum \frac{a}{a^2(a+b+c)+b^3+c^3}$$

$$= \sum \frac{a}{(a^3+b^3+c^3)+a^2b+a^2c}$$

AM–GM $\leq \sum \frac{a}{3abc+a^2b+a^2c}$

$$= \sum \frac{1}{3bc+ab+ac}$$

AM–GM $\leq \sum \frac{1}{5\sqrt[5]{a^2b^4c^4}} \times \frac{abc}{abc}$

$$= \sum \frac{\sqrt[5]{a^3bc}}{5abc}$$

AM–GM $\leq \sum \frac{((3a+b+c)15)}{5abc}$

$$\leq \frac{1}{5abc}\sum \frac{3a+b+c}{5}$$

but $\sum 3a+b+c = 5(a+b+c) = 5$

Hence LHS $\leq \frac{1}{5abc}$

4. Consider the following 3 × 2 array formed by using the numbers 1, 2, 3, 4, 5, 6

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a 3 × k array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers 7, 8, 9, ... , 3k such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \text{ and } \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

Sol. Note 1 + 6 = 2 + 5 = 3 + 4 = 7

and $1^2 + 6^2 = 37$

$2^2 + 5^2 = 29$

$3^2 + 4^2 = 25$

$(6k + 1)^2 + (6k + 6)^2 = 72k^2 + 14k + 37 \rightarrow$ (A)

$(6k + 2)^2 + (6k + 5)^2 = 72k^2 + 14k + 29 \rightarrow$ (B)

$(6k + 3)^2 + (6k + 4)^2 = 72k^2 + 14k + 25 \rightarrow$ (C)

So let us call pairs of type A, B, C clearly pairs of types A, B, C contributes 37, 29 and 25 respectively in sum of square and their contribution in sum is equal.

So we need equal number of pairs of each type in each row

⇒ Number of columns must be divisible by 6.

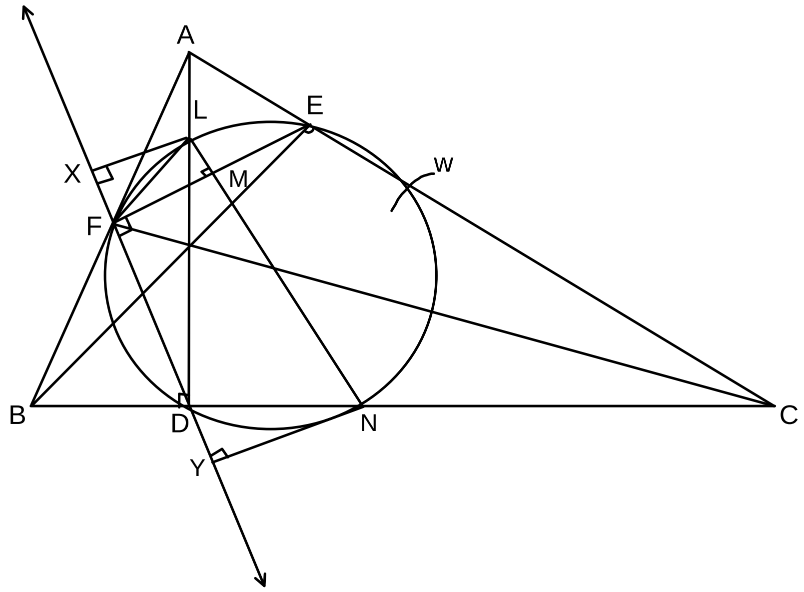
There can be many combinations for e.g.

$$\begin{pmatrix} 1 & 6 & 8 & 11 & 15 & 16 \\ 2 & 5 & 9 & 10 & 13 & 18 \\ 3 & 4 & 7 & 12 & 14 & 17 \end{pmatrix} \rightarrow \begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix}$$

Works

5. In an acute angled triangle ABC, let H be the orthocentre, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC respectively. Let X, Y be feet of altitudes from L, N on to the line DF. Prove that XM is perpendicular to MY.

Sol.



We know that w is nine pt circle and L, N are midpoints of arcs EF. So it makes LN a diameter and also perpendicular bisector of EF

$\therefore$ L-M-N and LN $\perp$EF

$\Rightarrow$□ LXFM, □NMFY are cyclic.

$\therefore \angle MXF = \angle MLF$

and$\angle MYF = \angle MNF$

$\therefore \angle MXF + \angle MYF = \angle MLF + \angle MNF$

$= 180 - \angle LFN = 90^o$

$\therefore$ In Δ XMY, third angle $\angle XMN = 90^o$

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that gcd (a, b) = gcd (b, c) = gcd (c, d) = gcd (d, a) = 1.

Sol. We will try to prove by induction
The smallest k for which

Total number pairs $\binom{k}{2} \geq 5k+1$

$$\frac{k(k-1)}{2} \geq 10k+2$$

$k^2 - 11k - 2 \geq 0$

$\Rightarrow k \geq 12$

For k = 12 there are $\binom{k}{2}$ = 66 pairs in total.

If at least 61 pairs are co-prime

$\Rightarrow$ We can easily find 4 numbers with (a, b) = (b, c) = (c, d) = (d, a) = 1

Let us assume the result for same k that for any k position integers if at least 5k + 1pairs are co-prime

Their exists, some 4 such that (c, b) = (b, c) = (c, d) = (d, a) = 1

Consider some (k + 1) positive integers such that at least (5k + 6) pairs are co-prime.

Clearly one of them say a must have at most 5 co-prime pairs.

Hence if we remove a and it's 5 pairs we get k position integers with (5k + 1) pairs.

So we get the result by induction.