IIT – JEE: 2019 **TW TEST (3 YRS)** **DATE: 17/08/18**

TOPIC: DEFINTE INTEGRATION (SOLUTION)

1. **(D)**

We have, $\int_{2}^{-4}\left(3-f\left(x\right)\right)dx=7$

$\Rightarrow \quad \int_{2}^{-4}3\,dx-\int_{2}^{-4}f\left(x\right)dx=7$

$\Rightarrow \quad 3\left(-4-2\right)-\int_{2}^{-4}f\left(x\right)dx=7$

$\Rightarrow \quad \int_{2}^{-4}f\left(x\right)dx=-25\Rightarrow\int_{-4}^{2}f\left(x\right)dx=25$

$\Rightarrow \quad \int_{-4}^{-1}f\left(x\right)dx+\int_{-1}^{2}f\left(x\right)dx=25$

$\Rightarrow \quad -4+\int_{-1}^{2}f\left(x\right)dx=25 \qquad \left[\because\int_{-1}^{-4}f\left(x\right)dx=4\right]$

$\Rightarrow \quad \int_{-1}^{2}f\left(x\right)dx=29$

$\Rightarrow \quad -\int_{1}^{-2}f\left(-t\right)dt=29$, where $t=-x$

$\Rightarrow \quad \int_{-2}^{1}f\left(-t\right)dt=-29\Rightarrow\int_{-2}^{1}f\left(-x\right)dx=-29$

2. **(C)**

We have, $f\left(x\right)=\dfrac{e^{x}}{1+e^{x}}$

$$\Rightarrow \quad f\left(a\right)+f\left(-a\right)=\frac{e^{a}}{1+e^{a}}+\frac{e^{-a}}{1+e^{-a}}$$

$$=\frac{e^{a}}{1+e^{a}}+\frac{1}{1+e^{a}}=1$$

Now, $\quad I_1=\int_{f(-a)}^{f(a)}x\,g\left\{x\left(1-x\right)\right\}dx$

$I_1=\int_{f(-a)}^{f(a)}\left(1-x\right)g\left\{\left(1-x\right)\left(1-\left(1-x\right)\right)\right\}dx$

$\left[\because f\left(a\right)+f\left(-a\right)=1,\int_{a}^{b}f\left(x\right)dx=\int_{a}^{b}f\left(a+b-x\right)dx\right]$

$I_1=\int_{f(-a)}^{f(a)}g\left\{\left(1-x\right)x\right\}dx-I_1$

$2I_1=I_2\Rightarrow I_2/I_1=2$

3. **(C)**

Let $\quad I=\int_{-\pi/2}^{\pi/2}\dfrac{e^{|\sin x|}\cos x}{1+e^{\tan x}}dx \qquad \ldots(i)$

$\therefore \quad I=\int_{-\pi/2}^{\pi/2}\dfrac{e^{|\sin x|}\cos\left(-x\right)}{1+e^{\tan x}}dx$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^{|\sin x|} \cos x}{1+e^{-\tan x}} dx \qquad \ldots \text{(ii)}$$

On adding eqs.(i) and (ii) we get

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \left(\frac{e^{|\sin x|} \cos x}{1+e^{\tan x}} + \frac{e^{|\sin x|} \cos x}{1+e^{-\tan x}} \right) dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} e^{|\sin x|} \cos x \, dx = 2I = 2\int_{0}^{\pi/2} e^{|\sin x|} \cos x \, dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi/2} e^{\sin x} \cos x \, dx = \left[e^{\sin x} \right]_0^{\pi/2} = e - 1$$

4. (C)

Given, $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for} |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$

$$\therefore \quad \int_{-2}^{3} f(x) dx = \int_{-2}^{2} f(x) dx + \int_{2}^{3} f(x) dx$$

$$= \int_{-2}^{2} e^{\cos x} \sin x \, dx + \int_{2}^{3} 2 \, dx$$

$$= 0 + 2[x]_2^3 \quad \left[\because e^{\cos x} \sin x \text{ is an odd function} \right]$$

$$= 2[3-2] = 2$$

5. (C)

Let $\quad I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \qquad \ldots \text{(i)}$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx = \int_{-\pi}^{\pi} \frac{a^x}{1+a^x} \cos^2 x \, dx \qquad \ldots \text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-\pi}^{\pi} \left(\frac{1+a^x}{1+a^x} \right) \cos^2 x \, dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x \, dx = 2\int_{0}^{\pi} \cos^2 x \, dx$$

$$= 2\int_{0}^{\pi} \frac{1+\cos 2x}{2} dx = \int_{0}^{\pi} (1+\cos 2x) dx$$

$$= \left[x + \frac{\sin 2x}{2} \right]_0^{\pi} = \pi \Rightarrow I = \frac{\pi}{2}$$

6. (C)

Let $\quad I = \int_{0}^{\pi} \frac{x}{1+\cos\alpha \sin x} dx \qquad \ldots \text{(i)}$

$$\Rightarrow \quad I = \int_{0}^{\pi} \frac{\pi - x}{1+\cos\alpha \sin(\pi - x)} dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi} \frac{\pi - x}{1+\cos\alpha \sin x} dx \qquad \ldots \text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_{0}^{\pi} \frac{dx}{1+\cos\alpha \sin x}$$

$$\Rightarrow \quad 2I = \pi \int_{0}^{\pi} \frac{\sec^2 x/2}{1+\tan^2 x/2 + 2\cos\alpha \tan x/2} dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2\,dt$

$\therefore \qquad 2I = \pi\int_0^\infty \frac{2dt}{1+t^2+2t\cos\alpha}$

$\Rightarrow \qquad I = \pi\int_0^\infty \frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$

$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t\cos\alpha}{\sin\alpha}\right)\right]_0^\infty$

7. (C)

Let $I = \int_{-2}^{0}\left[x^3+3x^2+3x+3(x+1)\cos(x+1)\right]dx$

$= \int_{-2}^{0}\left[(x+1)^3+2+(x+1)\cos(x+1)\right]dx$

Put $x+1 = t \Rightarrow dx = dt$

$\therefore \qquad I = \int_{-1}^{1}\left[t^3+2+t\cos t\right]dt$

$= \int_{-1}^{1}t^3\,dt + 2\int_{-1}^{1}dt + \int_{-1}^{1}t\cos t\,dt$

$= 0 + 2.2[x]_0^1 + 0 \qquad \left[\because t^3, t\cos t \text{ are odd function}\right]$

$= 4$

8. (D)

Let $\quad f(x) = \frac{x^7-3x^5+7x^3-x}{\cos^2 x}$

Now, $\quad f(-x) = -f(x)$. So, $\int_{-1}^{1}f(x)\,dx = 0$

Let $\quad g(x) = \cos^{-1}x$, then

$I_1 = \int_{-1}^{1}\cos^{-1}x\,dx = \int_{-1}^{1}\cos^{-1}(-x)\,dx$

$= \int_{-1}^{1}\left(\pi - \cos^{-1}x\right)dx = 2\pi - I_1$

Thus, $\quad I = I_1 = \pi$

9. (D)

For $0 < x < 1$, we have $\frac{1}{2}x^2 < x^2 < x$

$\Rightarrow \qquad -x^2 > -x \Rightarrow e^{-x^2} > e^{-x}$

Hence, $\int_0^1 e^{-x^2}\cos^2 x\,dx > \int_0^1 e^{-x^2}\cos^2 x\,dx$

Also, $\cos^2 x \le 1$

Therefore, $\int_0^1 e^{-x^2}\cos^2 x\,dx \le \int_0^1 e^{-x^2}dx < \int_0^1 e^{-x^2/2}\,dx$

$\therefore \qquad I = I_4$

10. (ABC)

Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx \qquad \ldots \text{(i)}$

$\therefore \qquad I=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}dx$

$$=\int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}dx \qquad \ldots \text{(ii)}$$

On adding Eqs., (i) and (ii), we get

$2I=\int_a^b 1\,dx \Rightarrow 2I=[x]_a^b \Rightarrow 2I=b-a \Rightarrow I=\frac{b-a}{2}$

Hence, option (ABC) satisfy the above condition

11. (ABD)

We have $f(2-x)=f(2+x), f(4-x)=f(4+x)$ or

$f(4+x)=f(4-x)=f(2+(2-x))=f(2-(2-x))=f(x)$

Thus the period of $f(x)$ is 4,

$\int_0^{50} f(x)dx=\int_0^{48} f(x)dx+\int_{48}^{50} f(x)dx=12\int_0^4 f(x)dx+\int_0^2 f(x)dx$

[In second integral, replacing x by x + 48 and then using $f(x)=f(x+48)$]

$$=12\left[\int_0^2 f(x)dx+\int_0^2 f(4-x)dx\right]+5$$

$$=12\left[\int_0^2 f(x)dx+\int_0^2 f(4-x)dx\right]+5$$

$$=24\int_0^2 f(x)dx+5=24\times 5+5=125$$

Also, $\int_{-4}^{46} f(x)dx=\int_{-4}^{-2} f(x)dx+\int_{-2}^{-2+48} f(x)dx$

$$=\int_0^2 f(x+4)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(x)dx+24\int_0^2 f(x)dx$$

$$=5+24\times 5=125$$

Again, $\int_2^{52} f(x)dx=\int_2^4 f(x)dx+\int_4^{4+48} f(x)dx$

$$=\int_2^2 f(4-x)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(4+x)dx+24\int_0^2 f(x)dx$$

$$=\int_0^2 f(x)dx+24\int_0^4 f(x)dx$$

$$=5+24\times 5=125$$

Again, $\int_1^{51} f(x)dx=\int_1^3 f(x)dx+\int_3^{3+48} f(x)dx$

$$=\int_1^3 f(x)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(x+1)dx+24\int_0^2 f(x)dx\neq 125$$

12. (B)

13. (C)

Solution

$f(x)=\sin x+\sin x\int_{-\pi/2}^{\pi/2} f(t)dt+\cos\times\int_{-\pi/2}^{\pi/2} t\,f(t)dt$

$$= \sin x\left(1+\int_{-\pi/2}^{\pi/2} f(t)\,dt\right)+\cos \times \int_{-\pi/2}^{\pi/2} t\, f(t)\,dt$$

$$= A\sin x + B\cos x$$

Thus, $A = 1+\int_{-\pi/2}^{\pi/2} f(t)\,dt = 1+\int_{-\pi/2}^{\pi/2}(A\sin t + B\cos t)\,dt$

$$= 1 + A\int_{-\pi/2}^{\pi/2}\sin t\,dt + B\int_{-\pi/2}^{\pi/2}\cos t\,dt$$

$$= 1 + A\times 0 + 2B\int_0^{\pi/2}\cos t\,dt = 1+2B[\sin t]_0^{\pi/2}$$

$\therefore \quad A = 1+2B$ … (i)

$$B = \int_{-\pi/2}^{\pi/2} t\, f(t)\,dt = \int_{-\pi/2}^{\pi/2} t(A\sin t + B\cos t)\,dt$$

$$= A\int_{-\pi/2}^{\pi/2} t\sin t\,dt + B\int_{-\pi/2}^{\pi/2} t\cos t\,dt$$

$$= 2A\int_0^{\pi/2} t\sin t\,dt + 0 = 2A[-t\cos t + \sin t]_0^{\pi/2}$$

$\therefore \quad B = 2A$ … (ii)

From Eqs. (i) and (ii), we get

$$A = -\frac{1}{3} \text{ and } B = -\frac{2}{3}$$

$\therefore \quad f(x) = -\frac{1}{3}(\sin x + 2\cos x)$

Now, $f(x) = -\frac{1}{3}(\sin x + 2\cos x)$

$$= \frac{-\sqrt{5}}{3}\sin\left(x+\tan^{-1}2\right) = \frac{-\sqrt{5}}{3}\cos\left(x-\tan^{-1}\frac{1}{2}\right)$$

Thus, the range of $f(x)$ is $\left[\frac{-\sqrt{5}}{3},\frac{\sqrt{5}}{3}\right]$

$\therefore f(x)$ is invertible , if

$$-\pi \le x-\tan^{-1}\frac{1}{2} \le 0$$

$\Rightarrow \quad -\frac{\pi}{2}-\tan^{-1}2 \le x \le \frac{\pi}{2}-\tan^{-1}2$

$\Rightarrow \quad 0 \le x-\tan^{-1}\frac{1}{2} \le \pi$

$\Rightarrow \quad \tan^{-1}\frac{1}{2} \le x \le \pi+\tan^{-1}\frac{1}{2}$

$\Rightarrow \quad \pi \le x-\tan^{-1}\frac{1}{2} \le 2\pi$

$\Rightarrow \quad x\in\left[\pi+\cot^{-1}2, 2\pi+\cot^{-1}2\right]$

Again, $\int_0^{\pi/2} f(x)\,dx = -\frac{1}{3}\int_0^{\pi/2}(\sin x + 2\cos x)\,dx$

$$= -\frac{1}{3}[-\cos x + 2\sin x]_0^{\pi/2} = -\frac{1}{3}[2+1]$$

$$= -1$$

14. **(B)**

$f(x)$ is an odd function. Thus,

$f(x)=-f(-x);\phi(-x)=\int_a^{-x} f(t)\,dt$

Put $t=-y$

$\therefore \quad \phi(-x)=\int_{-a}^{x} f(-t)(-dt)=\int_{-a}^{x} f(t)\,dt$

$=\int_{-a}^{a} f(t)\,dt+\int_a^x f(t)\,dt=0+\int_a^x f(t)\,dt=\phi x$

15. **(D)**

If $f(x)$ is an even function, then

$\phi(-x)=-\int_{-a}^{x} f(t)\,dt=-\int_{-a}^{a} f(t)\,dt-\int_a^x f(t)\,dt$

$=-2\int_0^a f(t)\,dt=\int_a^x f(t)\,dt$ [as $f(x)$ is an even function]

Now, $\int_0^a f(t)\,dt=0\Rightarrow\phi(-x)=-\int_a^x f(t)\,dt=-f(x)$

16. **(5)**

We have, $f(2x)=3f(x)$ …(i)

And $\int_0^1 f(x)\,dx=1$ … (ii)

From Eqs. (i) and (ii), we get $\frac{1}{3}\int_0^1 f(2x)\,dx=1$

Put $2x=t \Rightarrow 2dx=dt$

Then, $\frac{1}{6}\int_0^2 f(t)\,dt=1\Rightarrow\int_0^2 f(t)\,dt=6$

$\Rightarrow \quad \int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt=6\Rightarrow\int_1^2 f(t)\,dt=6-\int_0^1 f(t)\,dt$

$\Rightarrow \quad \int_1^2 f(t)\,dt=6-1\Rightarrow\int_1^2 f(t)\,dt=5$

17. **(4)**

$I_1=\int_0^1 x^{1004}(1-x)^{1004}\,dx$

$=2\int_0^{1/2} x^{1004}\left(1-x^{2010}\right)^{1004}\,dx$

Now, $I_2=\int_0^1 x^{1004}\left(1-x^{2010}\right)^{1004}\,dx$

Put $x^{1005}=t$ or $1005x^{1004}dx=dt$

$\therefore \quad I_2=\frac{1}{1005}\int_0^1 (t(2-t))^{1004}\,dt$

$=\frac{1}{1005}\int_0^1 (t(x-t))^{1004}\,dt=\frac{1}{1005}\int_0^1 t^{1004}(2-t)^{1004}\,dt$

Now put $t=2y$ or $dt=2dy$

$\therefore \quad I_2=\frac{1}{1005}\int_0^{1/2}(2y)^{1004}(2-2y)^{1004}\,2dy$

$=\frac{1}{1005}2.2^{1004}.2^{1004}\int_0^{1/2} y^{1004}(1-y)^{1004}\,dy$

$=\frac{1}{1005}.2^{2009}\int_0^{1/2} y^{1004}(1-y)^{1004}\,dy$

$$= \frac{2^{2008}}{1005} I_1$$

$$\therefore \qquad \frac{I_1}{I_2} = \frac{1005}{2^{2008}} \Rightarrow \frac{2^{2010}}{1005} \cdot \frac{I_1}{I_2} = 4$$

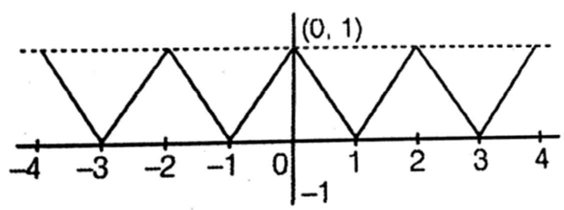
18. **(3)**

We have, $f(x) = \begin{cases} \left| x - [x] \right|, & \text{when } [x] \text{ is odd} \\ \left| x - [x] - 1 \right|, & \text{when } [x] \text{ is even} \end{cases}$

$$= \begin{cases} \left| x - [x] \right|, & \text{when } [x] \text{ is odd} \\ \left| \{x\} - 1 \right|, & \text{when } [x] \text{ is even} \end{cases}$$

$$= \begin{cases} \{x\}, & \text{when } [x] \text{ is odd} \\ 1 - \{x\}, & \text{when } [x] \text{ is even} \end{cases}$$

The graph of $f(x)$ is as below



$$\therefore \int_{-2}^{4} f(x)\,dx = 6\int_{-2}^{-1} f(x)\,dx = 6 \times \frac{1}{2} = 3$$

19. **(8)**

Let $g(x) = x^3 f(x)$

Then, $g(-x) = (-x)^3 t(-x)$

$$\Rightarrow \qquad g(-x) = -x^3 \left(\frac{e^{-x}+1}{e^{-x}-1} \right) = -x^3 \left(\frac{1+e^x}{1-e^x} \right)$$

$$= -x^3 \left[-\left(\frac{e^x+1}{e^x-1} \right) \right] = x^3 - \frac{e^x+1}{e^x-1} = g(x)$$

$\Rightarrow g(x)$ is an even function

Hence, $\int_{-1}^{1} t^3 f(t)\,dt = 1\int_{0}^{1} t^3 f(t)\,dt = 2 \times 4 = 8$

20. **(6)**

We have, $I = \int_1^4 \log_e [x]\,dx$

$$= \int_1^2 \log_e [x]\,dx + \int_2^3 \log_e [x]\,dx + \int_3^4 \log_e [x]\,dx$$

$$= \int_1^2 \log_e 1\,dx + \int_2^3 \log_e 2\,dx + \int_3^4 \log_e 3\,dx$$

$$= 0 + \log_e 2 [x]_2^3 + \log_e 3 [x]_3^4$$

$$= \log_e 2 + \log_e 3 = \log_e 6$$

$$\therefore \qquad \lambda = 6$$