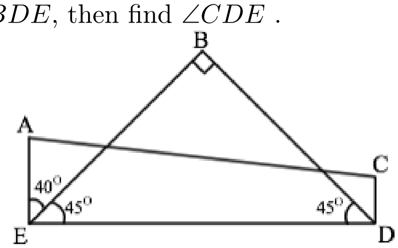
NMTC VII,VII-2018
Kaprekar contest -Sub Junior Level

November 5, 2018

1. A lucky year is one in which at least one date, which written in the from day/month/year, has the following property:
   The product of the month times the day equals the last two digits of the year. For example 1956 is lucky year because it has the date 7/8/56 where 7x8 = 56 but 1962 is not a lucky year as 62 = 62x1 or 31x 2 but 31/2/1962 is not a valid date.
   From 1900 to 2018 how many years are not lucky(not including 1900 and 2018)?
   Solution:
   The possible values for dates are from set $\{1, 2, 3, \cdots, 31\}$
   The possible values for months are from the set $\{1, 2, \cdots, 12\}$
   Note all numbers till 31 will be lucky there there is no unlucky year after 2000
   Also all primes greater than 31 are unlucky along with few numbers like 56 so we have to do it trail and error wise .
   The possibilities for non lucky years are as follows
   $1937, 1941, 1943, 1947, 1953, 1958$(as it is not leap year),
   $1959, 1961, 1962, 1967, 1971, 1973,$
   $1974, 1979, 1982, 1983, 1986, 1989,$
   $1994, 1997, 2000$
   So totally there are 21 unlucky years .

2. In the figure $\angle A, \angle B, \angle C$are right angles If $\angle AEB = 40$ and $\angle BED = \angle BDE$, then find $\angle CDE$.



   Solution: From $\triangle BED$ we know that $\angle EBD + \angle BED + \angle BDE = 180$

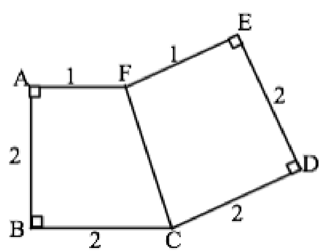
$\angle BED + \angle BDE = 90$

$$\therefore \angle BED = \angle BDE = 45$$

As the sum of all angles of the quadrilateral is 360 .
We have $\angle A + \angle AED + \angle CDE + \angle C = 360$
$\therefore \angle A + \angle AEF + \angle BED + \angle CDE + \angle C = 360$
$\therefore 90 + 40 + 45 + \angle CDE + 90 = 360$

$$\therefore \angle CDE = 95$$

3. a) $ABCDEF$ is a hexagon in which $AB = BC = CD = DE = 2$ and $EF = FA = 1$. Its interior angle $C$is between 90 and 180 and F is greater than 90 . The rest of the angles is 90. What is its area ?



Sol: Only important part is to draw proper figure.
Now its easy to see that $\square ABCF$ and $\square FCDE$ are trapeziums
$\therefore A[\square ABCF] = \frac{1}{2}AB(AF + BC) = \frac{1}{2}(2)(3) = 3unit^2$
$\therefore A[\square FCDE] = \frac{1}{2}ED(FE + DC) = \frac{1}{2}(2)(3) = 3unit^2$
$\therefore A[Hexagon ABCDEF] = 6unit^2$

b) A convex polygon with n sides has all angles equal to 150 ,except one angle. List all possible values of n.
Solution: As polygon is convex each angle is less than 180.
The the last angle be θ
The sum of all angles is $180(n-2) = 150(n-1) + \theta$
$\therefore \theta = 180(n-2) - 150(n-1) = 30n - 210 < 180$
$\therefore 30n < 210 + 180$
$\therefore n < 7 + 6 = 13$
Also $\theta + 210 = 30n$
$\therefore \frac{\theta}{30} + 7 = n < 13$
$\therefore 7 < n < 13$ so $n \in \{8, 9, 10, 11, 12\}$
But for $n = 12$ we get $\theta = 150$ so $n \in \{8, 9, 10, 11\}$

4. a, b, c are distinct nonzero reals such that $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c}$. Find all possible values of $a^3 + b^3 + c^3$.
Solution: Let $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c} = k$.
$\therefore a^3 - ka + 1 = 0$

$\therefore b^3 - kb + 1 = 0$
$\therefore c^3 - kc + 1 = 0$
Consider the cubic equation $x^3 - kx + 1 = 0$, then a, b, c , are the roots of this equation.
Hence the sum of the roots $= a + b + c = 0$.
And the product of roots $abc = -1$
Consider the identity $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
$\therefore$ As $a + b + c = 0$
$\therefore a^3 + b^3 + c^3 = 3abc = (-3)$

5. Find the smallest positive integer with exactly 100 different factors including 1 and the number itself.
Solution:
The number of factors of number $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is given by

$$\tau(n) = (1 + a_1)(1 + a_2) \cdots (1 + a_k)$$

So by trial and error on different factors of 100 we get the smallest value of n for smallest possible primes with highest powers.
$\tau(n) = 100 = 2 \cdot 2 \cdot 5 \cdot 5 = (1 + 1)(1 + 1)(1 + 4)(1 + 4)$
So smallest possible number will be $2^4 \cdot 3^4 \cdot 5 \cdot 7 = 45360$

6. a) What is the sum of the digits of the smallest positive integer which is divisible by 99 and has all its digits equal to 2?
Solution:
As the number is divisible by 9 the sum of the digits is also divisible by 9
Let number of the digits is odd say $2n + 1$ then the (sum of the odd places)-(sum of even places)=2
So in order that the number is divisible by 11, it must have even number of digits say $2n$.
Hence (sum of the odd places)-(sum of even places)=0 and it will be divisible by 11.
Hence the smallest possible number will have 18 digits .
Hence the sum of the digits is 36.

b) When 270 is divided by the odd number n the quotient is prime number and the reminder is zero. What is n?
Solution: Hence n divides 270 and other factor is prime say p.
$\therefore 270 = 3^3$x2x5 =nxp where n is odd so clearly $p = 2$
$\therefore n = 135$

7. Consider the sum
$A = \frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \cdots + \frac{1}{99\cdot100}$ and $B = \frac{1}{51\cdot100} + \frac{1}{52\cdot99} + \cdots + \frac{1}{100\cdot51}$
Express $\frac{A}{B}$ as an irreducible fraction.
Solution:

$A = \frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \cdots + \frac{1}{99\cdot100} = \frac{2-1}{1\cdot2} + \frac{4-3}{3\cdot4} + \cdots + \frac{100-99}{99\cdot100}$
$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100}$
$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{99} + \frac{1}{100} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{100}\right)$
$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{99} + \frac{1}{100} - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50}\right)$
$A = \frac{1}{51} + \frac{1}{52} + \frac{1}{53} + \cdots + \frac{1}{99} + \frac{1}{100}$
Also $151B = \frac{151}{51\cdot100} + \frac{151}{52\cdot99} + \cdots + \frac{151}{100\cdot51}$
$151B = \frac{100+51}{51\cdot100} + \frac{99+52}{52\cdot99} + \cdots + \frac{100+51}{100\cdot51} = \frac{1}{51} + \frac{1}{100} + \frac{1}{52} + \frac{1}{99} + \cdots + \frac{1}{100} + \frac{1}{51}$
$\therefore 151B = 2\left(\frac{1}{51} + \frac{1}{52} + \frac{1}{53} + \cdots + \frac{1}{99} + \frac{1}{100}\right)$
$\therefore 151B = 2A$
$\therefore \frac{A}{B} = \frac{151}{2}$

8. Let a, b, c be real numbers , not all of them are equal. Prove that $a+b+c = 0$ then $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$
Prove the converse if $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$, then $a + b + c = 0$
Solution:
If $a + b + c = 0$ so $a = -(b + c)$
$\therefore a^2+ab+b^2 = (b + c)^2 - (b+c)b+b^2 = b^2+2bc+c^2-b^2-bc+b^2 = b^2+bc+c^2$
Similarly by taking $b = -(c+a)$we get the result $b^2+bc+c^2 = c^2+ca+a^2$
.
$\therefore a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$
Conversely
If $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$
$a^2 + ab + b^2 = b^2 + bc + c^2$
$\therefore a^2 - c^2 + ab - bc = 0$
$\therefore a^2 - c^2 + ab - bc = (a - c)(a + c) + b(a - c) = (a - c)(a + b + c) = 0$
As $a \neq c$ we get $a + b + c = 0$