STD : XII **PRELIMINARY EXAMINATION** **MARKS: 80**

DATE : **MATHEMATICS & STATISTICS** **TIME : 3 hrs**

SOLUTION set I

Section – I

Q.1 (A) Select the correct answer : **(6 Marks)**

1. (d) $(\sim p \vee \sim q) \to p$
2. (c) $\prod$ /6
3. (b) 0 , 1 , 2

Q.1 (B) Attempt any Three : **(6 Marks)**

1. Find the values of K if the lines represented by $K(x^2+y^2)=8xy$ are co-incident.

Ans. $Kx^2-8xy+Ky^2=0$

Comparing with $ax^2-2hxy+by^2=0$

Here $a=K,\ 2h=-8,\ b=k$

$\therefore$ Lines are coincident

$\therefore h^2-ab=0$ (1 mark)

$\therefore (-4)^2\ K.K=0$

$\therefore K^2=16$

$\therefore K=\pm 4$ (1 mark)

2. If $A=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$ and B = $\begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}$ then find (AB)$^{-1}$

$AB=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}$

$=\begin{bmatrix}11 & 3\\7 & 2\end{bmatrix}$ (1 mark)

$|AB|=\begin{vmatrix}11 & 3\\7 & 2\end{vmatrix}=1\neq 0$

Hence , (AB)$^{-1}$ exists

$\therefore (AB)^{-1}=\dfrac{1}{|AB|}\ adj\ AB=\begin{bmatrix}2 & -3\\-7 & 11\end{bmatrix}$ (1 mark)

3. Find the angle between the planes $\bar{r}.(2\bar{i}-\bar{j}+\bar{k})=6$ and $\bar{r}.(\bar{i}-2\bar{j}+4\bar{k})=7$

Ans. Given vector equations of the planes are

$\bar{r}.(2\bar{i}-\bar{j}+\bar{k})=6$ ……………(i)

$\bar{r}.(\bar{i}-2\bar{j}+4\bar{k})=7$ …………….(ii)

Then normal to the planes (i) and (ii) are given by vectors

$\bar{n}_1=2\bar{i}-\bar{j}+\bar{k} \quad \therefore \left|\bar{n}_1\right|=\sqrt{6}$

$\bar{n}_2=\bar{i}-2\bar{j}+2\bar{k} \quad \therefore \left|\bar{n}_2\right|=\sqrt{21}$

Since angle θ between the two planes (i) and (ii) is given by formula

$$\cos\theta=\frac{\bar{n}_1.\bar{n}_2}{\left|\bar{n}_1\right|\left|\bar{n}_2\right|}$$ (1 mark)

$$=\frac{2(1)+(-1)(-2)+(1)(4)}{\sqrt{6}\sqrt{21}}$$

$$=\frac{8}{3\sqrt{14}}$$

$$\therefore \theta=\cos^{-1}\frac{8}{3\sqrt{14}}$$ (Ans) (1 mark)

4. Find the volume of the parallelepiped formed by the vectors $\bar{a}=\bar{i}+2\bar{j}+3\bar{k}$, $\bar{b}=2\bar{i}+\bar{j}+4\bar{k}$ and $\bar{c}=3\bar{i}+4\bar{j}+\bar{k}$.

Ans. Volume of parallelepiped

$=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$$=\begin{vmatrix}1 & 2 & 3\\ 2 & 1 & 4\\ 3 & 4 & 1\end{vmatrix}$$ (1 mark)

$=1(1-16)-2(2-12)+3(8-3)$

$=-15+20+15$

$=20$ cu. units (1 mark)

5 By vector method , find the equation of the line passing through the point A(2 , -3 , - 4) and parallel to OB vector where point O is origin and B is (2 , - 2 , - 1) .

Ans. Required line passes through point $A(2,-3,-4)$ i.e ; p.v. of point

$$\overline{OA} \equiv \bar{a} = 2\bar{i} - 3\bar{j} - 4\bar{k}$$

$$B = (2,-2,-1)$$

i.e. $\overline{OB} = \bar{b} = 2\bar{i} = 2\bar{j} - \bar{k}$ (1 mark)

Now equation of line through point A $(\bar{a})$ and parallel to $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \bar{r} = (2\bar{i} - 3\bar{j} - 4\bar{k}) + \lambda(2\bar{i} - 2\bar{j} - \bar{k})$$

$$\therefore \bar{r} = (2+2\lambda)\bar{i} - (3+2\lambda)\bar{j} - (4+\lambda)\bar{k}$$ (1 mark)

is the required equation of line in vector form.

Now $x\bar{i} + y\bar{j} + z\bar{k} = (2+2\lambda)\bar{i} - (3+2\lambda)\bar{j} - (4+\lambda)\bar{k}$

$\therefore x = 2+2\lambda, \quad y = -3-2\lambda, \quad z = -4-\lambda$

$\therefore$ Equation of line in symmetric form is as

$$\frac{x-2}{2} = \frac{y+3}{-2} = \frac{z+4}{-1}$$ (1 mark)

Q.2 (A) Attempt any Two : **(6 marks)**

1. Determine whether the following statement pattern is a 'tautology' or a 'contradiction' or 'neither' of the two. $(\sim p \vee q) \rightarrow p \wedge (q \vee \sim p)$

Ans.

			A			B	
p	q	$\sim p$	$\sim p \vee q$	$\sim q$	$q \vee \sim p$	$p \wedge (q \vee \sim p)$	$A \rightarrow B$
1	2	3	4	5	6	7	8
T	T	F	T	F	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	T	F	F
F	F	T	T	T	T	F	F

A,B ------(1 mark)

$A \rightarrow B$---(1 mark)

From column 8, truth values of the given statement pattern are some T's and some F's. Hence the given statement is contingency. (1 mark)

2. Find ‘p’ if the vectors $\bar{i}+p\bar{j}-3\bar{k},\ 2\bar{i}+\bar{j}-4\bar{k}$ and $\bar{i}-\bar{j}+\bar{k}$ are co-planar.

Ans. Let $\bar{a}=\bar{i}+p\bar{j}-3\bar{k}, \bar{b}=2\bar{i}+\bar{j}-4\bar{k}$, $\bar{c}=\bar{i}-\bar{j}+\bar{k}$

Since the vectors are coplanar

$\therefore \left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$ (1 mark)

i.e. $\begin{vmatrix} 1 & p & -3 \\ 2 & 1 & -4 \\ 1 & -1 & 1 \end{vmatrix}=0$ (1 mark)

i.e. $1(1-4)-p(2+4)-3(-2-1)=0$ (1 mark)

$\therefore\ -6p+6=0 \Rightarrow 6p=6$

$\Rightarrow p=1$

3. If θ is the measure of acute angle between the lines represented by $ax^2+2hxy+by^2=0$, then show that $\tan\theta=\left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$

Ans. Given homogeneous equation is

$ax^2+2hxy+by^2=0$ ….. (1)

Let m be the slope of the lines given by (1)

$\therefore bm^2+2hm+a=0$ …..(2)

is the auxiliary equation of (1), which is quadratic in $'m'$ having two roots say m_1 and m_2, the slopes of the lines represented by (1)

$\therefore\ m_1+m_2=-\dfrac{2h}{b}$, and $m_1m_2=\dfrac{a}{b}$ (1 mark)

Now $(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$

$=\left(\dfrac{-2h}{b}\right)^2-4\times\dfrac{a}{b}$

$=\dfrac{4(h^2-ab)}{b^2}$

$|m_1-m_2|=\dfrac{2\sqrt{h^2-ab}}{a+b}$

And $1+m_1+m_2=1+\dfrac{a}{b}=\dfrac{a+b}{b}$ …….(iii)

$\therefore (1+m_1m_2)=(m_1+m_2)^2-4m_1m_2$

$=\left(\dfrac{-2h}{b}\right)^2-4\left(\dfrac{a}{b}\right)$

$=\dfrac{4\left(h^2-ab\right)}{b^2}$

$\therefore |m_1-m_2|=\left|\dfrac{2\sqrt{h^2-ab}}{b}\right|$ ……..(iv)

If θ is the acute angle between the lines given by equation (i)

Then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ (1 mark)

$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$ ……by (iii) and (iv) provided a + b $\neq$ 0 (1 mark)

Hence Proved

Q.2 (B) Attempt any Two : **(8 marks)**

1.Two food products A and B are to be purchased. Their contents and price per unit are given in the following table.

Product	A	B
Calories	2	3
Vitamins	2	1
Price	3	4

Minimum requirements of calories and vitamins are 36 and 14 units respectively. Formulate this problem as a L.P.P. to minimize the cost.

Ans. Let x = No. of units of product A

y = No. of units of product B (1 mark)

$\therefore x \geq 0, \quad y \geq 0$

Cost of product A per unit = Rs. 3

Cost of product B per unit = Rs. 4

$\therefore$ Cost function Z = 3x+4y (1 mark)

Now from the given data, the constraints are as follows :

$\therefore 2x + 3y \geq 36$

$2x + y \geq 14$

$x \geq 0, \quad y \geq 0$

$\therefore$ L.P.P. is formulated as

Maximum $z = 3x + 4y$ (1 mark)

Subject to $2x + 3y \geq 36$

$2x + y \geq 14$

$x \geq 0, \ y \geq 0$ (1 mark)

2. If $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$, then find A^{-1} by adjoint method.

Ans. $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix}$

$= 4(0-4) - 3(3-4) + 3(4-0)$

$= 4(-4) - 3(-1) + 3(4)$

$= -16 + 3 + 12 = -1$ (1 mark)

$\neq 0$

$\therefore A^{-1}$ exists.

Now,

$$C_{11} = (-1)^{1+1} = \begin{vmatrix} 0 & -1 \\ -4 & -3 \end{vmatrix} = -4$$

$$C_{12} = (-1)^{1+2} = \begin{vmatrix} -1 & -1 \\ -4 & -3 \end{vmatrix} = 1$$

$$C_{13} = (-1)^{1+3} = \begin{vmatrix} -1 & 0 \\ -4 & -4 \end{vmatrix} = 4$$

$$C_{21} = (-1)^{2+1} = \begin{vmatrix} 3 & 3 \\ -4 & -3 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2} = \begin{vmatrix} 4 & 3 \\ -4 & -3 \end{vmatrix} = 0$$

$$C_{23} = (-1)^{2+3} = \begin{vmatrix} 4 & 3 \\ -4 & -4 \end{vmatrix} = 4$$

$$C_{31} = (-1)^{3+1} = \begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} = -3$$

$$C_{32} = (-1)^{3+2} = \begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} = 1$$

$$C_{33} = (-1)^{3+3} = \begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$\therefore \text{ Cofactor matrix of A } = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$$ (1 mark)

$\therefore$ Adj.A = transpose of cofactor matrix of A.

$$= \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$ (1 mark)

$$\therefore A^{-1} = \frac{1}{|A|}.\, adj\ A$$

$$= \frac{1}{-1}\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$ (1 mark)

3. Find the general solution of $\sin x + \sin 3x + \sin 5x = 0$.

Ans. As given $\sin x + \sin 3x + \sin 5x = 0$

$\therefore \quad (\sin x + \sin 5x) + \sin 3x = 0$

$\therefore \quad 2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{5x-x}{2}\right) + \sin 3x = 0$

$\therefore \quad 2\sin 3x \cos 2x + \sin 3x = 0$ (1 mark)

$\therefore \quad \sin 3x(2\cos 2x + 1) = 0$

$\therefore \quad \sin 3x = 0$ or $2\cos 2x + 1 = 0$

$\therefore \quad \sin 3x = 0$ …….(i) or $\cos 2x = -\frac{1}{2}$ ….. (ii) (1 mark)

For (ii) $\cos 2x = -\cos\frac{\pi}{3}$

$\therefore \quad \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right)$ ……(by allied angles)

$\cos 2x = \cos\frac{2\pi}{3}$ (1 mark)

$\therefore$ from (i) and (ii) we get

$\therefore \quad \sin 3x = 0$ or $\cos 2x = \cos\frac{2\pi}{3}$

$\therefore \quad 3x = n\pi$, $n \in Z$ or $2x = 2m\pi \pm \frac{2\pi}{3}$, where $m \in Z$.

Hence, the required solutions is $\therefore x = \frac{n\pi}{3}$ or $x = m\pi \pm \frac{\pi}{3}$, where $n, m \in Z$. (1 mark)

Q.3 (A) Attempt any Two : **(6 marks)**

1. Find the values of p and q if the equation $12x^2 + 7xy - py^2 + 18x + qy + 6 = 0$ represents a pair of perpendicular lines.

Ans. Given equation is $12x^2 + 7xy - py^2 + 18x + qy + 6 = 0$ …..(1)

Comparing it with standard equation : we have $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 12,\ h = \frac{7}{2},\ b = -p,\ g = 9,\ f = \frac{q}{2}, c = 6$

Since, the lines are perpendicular to each other

$\therefore \quad a + b = 0$

$12 + (-p) = 0$

$\therefore \quad p = 12$

$\therefore \quad b = -12$ (1 mark)

Equation (1) represents pair of standard lines

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

i.e. $12(-12)6 + 2\left(\frac{q}{2}\right)(9)\left(\frac{7}{2}\right) - 12\left(\frac{q}{2}\right)^2 - (-12)(9)^2 - 6\left(\frac{7}{2}\right)^2 = 0$ (1 mark)

i.e. $-3q^2 + \frac{63}{2}q + 108 - \frac{147}{2} = 0$

i.e. $-6q^2 + 63q + 69 = 0$

i.e. $2q^2 - 21q - 23 = 0$

i.e. $(2q - 23)(q + 1) = 0$

i.e. $q = \frac{23}{2}$ or $q = -1$ (1 mark)

2. Without using the truth table, show that $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

Ans. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$ Conditional Law (1 mark)

$\sim p \leftrightarrow q \equiv \sim [(\sim p \vee q) \wedge (\sim q \vee p)]$

$\equiv \sim(\sim p \vee q) \vee \sim(\sim q \vee p)$ De Morgan's Law (1 mark)

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ De Morgan's Law (1 mark)

3. Find the Cartesian equation of the plane passing through the points (1, 1, 1), (2, 4, 3) and (5, 9, 7) using Vector methods.

Ans. Let A ≡ (1, 1, 1), B ≡ (2, 4, 3) and C ≡ (5, 9, 7) be the given points, whose P.V.s are $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.

$\therefore\ \bar{a} = \bar{i} - \bar{j} + \bar{k},\ \bar{b} = 2\bar{i} + 4\bar{j} + 3\bar{k}$ and $\bar{c} = 5\bar{i} - 9\bar{j} + 7\bar{k}$

$\therefore \bar{b} - \bar{a} = \bar{i} + 3\bar{j} + 2\bar{k}$ and $\bar{c} - \bar{a} = 4\bar{i} + 8\bar{j} + 6\bar{k}$

Let P($\bar{r}$) be any point in plane where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\therefore$ Equation of plane is given by $\left[\bar{r} - \bar{a} \quad \bar{b} - \bar{a} \quad \bar{c} - \bar{a}\right] = 0$ (1 mark)

i.e. $\begin{vmatrix} x-1 & y-1 & z-1 \\ 1 & 3 & 2 \\ 4 & 8 & 6 \end{vmatrix} = 0$ (1 mark)

$\Rightarrow (x-1)(18-16) - (y-1)(6-8) + (z-1)(8-12) = 0$

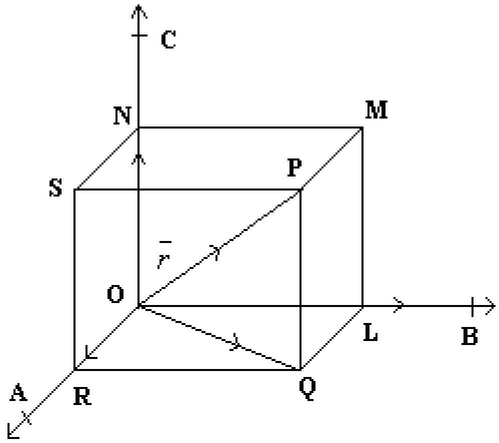
i.e. $x + y - 2z = 0$ (1 mark)

is the required equation of plane.

Q.3 (B) Attempt any Two : **(8 marks)**

1. If $\bar{a}, \bar{b}, \bar{c}$ are three non-zero, non coplanar vectors then prove that any vector $\bar{r}$ in the space can be uniquely expressed as a linear combination $x\bar{a}+y\bar{b}+z\bar{c}$, where x, y and z are scalars.

Ans. Let O be any point in space and $\overline{OA}=\bar{a}$, $\overline{OB}=\bar{b}$, $\overline{OC}=\bar{c}$, $\overline{OP}=\bar{r}$



Through the point P, draw the parallel planes AOB, BOC and COA completing the parallelopiped with OP as its diagonal.

Now $\overline{OR}$ and $\overline{OA}$ are collinear, therefore there exists scalar x such that $\overline{OR}=x.\overline{OA}=x.\bar{a}$

Similarly $\overline{OL}=y.\overline{OB}=y.\bar{b}$ (1 mark)

And $\overline{ON}=z.\overline{OC}=z.\bar{c}$

Where y and z are scalars.

By law of parallelogram $\overline{OQ}=\overline{OR}+\overline{OL}$

$=x\bar{a}+y\bar{b}$ ....(1)

Also, by law of triangle $\overline{OP}=\overline{OQ}+\overline{QP}$

$=\overline{OQ}+\overline{ON}$ $(\overline{QP}=\overline{ON})$

$=x\bar{a}+y\bar{b}+z\bar{c}$ (from 1)

$\therefore \bar{r}=x\bar{a}+y\bar{b}+z\bar{c}$ is a linear combination of $\bar{a},\bar{b}$ and $\bar{c}$. (1 mark)

Uniqueness : Let if possible

$\bar{r}=x\bar{a}+y\bar{b}+z\bar{c}=x'\bar{a}+y'\bar{b}+z'\bar{c}$

Suppose $x\neq x'$

Now $x\bar{a}+y\bar{b}+z\bar{c}=x'\bar{a}+y'\bar{b}+z'\bar{c}$

$\therefore (x-x')\bar{a}+(y-y')\bar{b}+(z-z')\bar{c}=0$

$\therefore (x-x')\bar{a}=-(y-y')\bar{b}-(z-z')\bar{c}=0$ (1 mark)

$\therefore \bar{a}=\left(\frac{y-y'}{x-x'}\right)\bar{b}-\left(\frac{z-z'}{x-x'}\right)\bar{c}$

$\because x\neq x'$

i.e. $x-x'\neq 0$

$\therefore$ $\bar{a}$ is a linear combination of $\bar{b}$ and $\bar{c}$ which is a contradiction to assumption $x\neq x'$ and the statement of theorem.

$\therefore x = x'$

Similarly, we can show that $y = y'$ and $z = z'$ (1 mark)

$\therefore \bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$ is unique.

Hence the theorem is proved.

2. Find the equation of a line in the Cartesian form passing through the point (3, 2, -1) and perpendicular to the vectors $3\bar{i} + 4\bar{j} + 5\bar{k}$ and $\bar{i} - \bar{j} + \bar{k}$.

Ans. Given point is $A \equiv (3, 2, -1)$ and given vectors are $\bar{b} = 3\bar{i} + \bar{j} + 5\bar{k}$ $\quad \bar{c} = \bar{i} - \bar{j} + \bar{k}$ and $P(x, y, z)$ be any point on the required line.

Let $\bar{n}$ be the vector perpendicular to $\bar{b}$ and $\bar{c}$

$$\therefore \quad n = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & -4 & 5 \\ 1 & -1 & 1 \end{vmatrix}$$ (1 mark)

i.e. $\bar{n} = \bar{i} + 2\bar{j} + \bar{k}$ (1 mark)

Now, $\overline{AP} \parallel \bar{n}$ as A and P lies in the same plane

$\therefore$ For a non-zero scalar λ

i.e. $\overline{AP} = \lambda.\bar{n}$

i.e. $(\bar{p} - \bar{a}) = \lambda.\bar{n}$ (1 mark)

i.e. $(x-3)\bar{i} + (y-2)\bar{j} + (z+1)\bar{k} = \lambda(\bar{i} + 2\bar{j} + \bar{k})$

i.e. $x - 3 = (1)\lambda,\ y - 2 = 2\lambda,\ z + 1 = (1)\lambda$

i.e. $$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+1}{1} = \lambda$$ (1 mark)

3. State and Prove Sine rule.

Ans.

Theorem : Sides of a triangle are proportional to the Sine of the opposite angles.

$\therefore$ In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, with the usual notations. (1 mark)

Proof : Consider any triangle ABC. As we know that each angle of a triangle cannot be obtuse, consider any of the angles say $\angle B$ to be acute.

Hence cases (i) $\angle C$ is acute (ii) $\angle C$ is obtuse (iii) $\angle C$ is a right angle.

(Refer figures respectively)

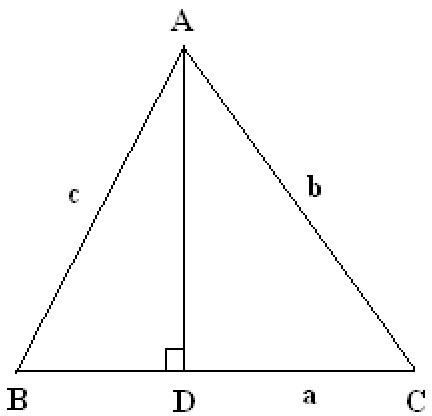


Fig.1

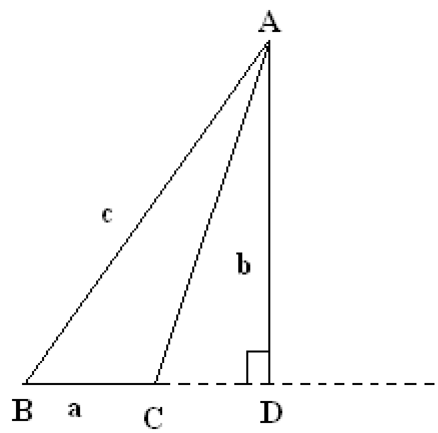


Fig.2

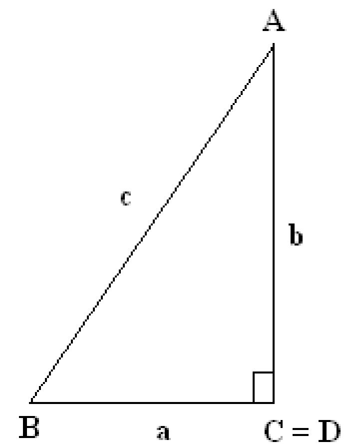


Fig.3

Now as shown in figures draw AD perpendicular to BC or BC produced. In each of the three cases we have

$$\frac{AD}{AB} = \sin B$$

$$\therefore \quad \frac{AD}{c} = \sin B$$

$$\therefore \quad AD = c\sin B \qquad \text{.......(i)} \qquad \text{(1 mark)}$$

Now in figure 1 $\quad \dfrac{AD}{AC} = \sin C$

$$\therefore \quad \frac{AD}{b} = \sin C$$

$$\therefore \quad AD = b\sin C$$

In figure 2 $\quad \dfrac{AD}{AC} = \sin(\pi - C)$

$$\therefore \quad \frac{AD}{AC} = \sin C$$

$$\therefore \quad \frac{AD}{b} = \sin C$$

$$\therefore \quad AD = b\sin C$$

In figure 3 $\quad \dfrac{AD}{AC} = 1 \quad = \sin\dfrac{\pi}{2}$

$$\therefore \quad \frac{AD}{AC} = \sin C$$

$$\therefore \quad \frac{AD}{b} = \sin C$$

$$\therefore \quad AD = b\sin C \qquad \text{.......(ii)} \qquad \text{(1 mark)}$$

From (i) and (ii) we get $\quad AD = c\sin B = b\sin C$

$$\therefore \quad \frac{c}{\sin C} = \frac{b}{\sin B}$$

Similarly we can get $\quad \dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

Hence $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Therefore, the Sine Rule is proved. (1 mark)

Section – II

Q. 4 (A) Select the correct answer : **(6 marks)**

1. (b) 0
2. (a) e
3. (a) 0.6,

Q. 4 (B) Attempt any Three : **(6 marks)**

1. Find the approximate value of $\tan^{-1}(0.999)$

Ans. Let $f(x) = \tan^{-1} x$

$$f'(x) = \frac{1}{1+x^2}$$

$$0.999 = 1 - 0.001$$
$$= a + h$$

where $a = 1$, $h = -0.001$

Now $f(a) = f(1) = \tan^{-1}(1) = \frac{\pi}{4}$

$$f'(a) = f'(1) = \frac{1}{1+(1)^2} = \frac{1}{2} = 0.5$$

By approximation formula

$$f(a+h) \doteq f(a) + hf'(a)$$ (1 marks)

$$\tan^{-1}(1-0.001) \doteq \frac{\pi}{4} + (-0.001)\ (0.05)$$

$$\therefore \tan^{-1}(0.999) \doteq \frac{\pi}{4} - 0.0005$$ (1 marks)

$$\doteq 0.7852$$

2. Form the differential equation by eliminating the arbitrary constant ‘a’ from the relation $(x-a)^2 + y^2 = 1$

Ans. Given $(x-a)^2 + y^2 = 1$ ……..(i)

Differentiate (i) w . r .t. x ,

$$2(x-a) + 2y.\frac{dy}{dx} = 0$$ (1 mark)

$\therefore$ $(x-a) = -y\frac{dy}{dx}$ ………(ii)

From (i) and (ii)

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = 1$$

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 1$$

$$y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = 1$$ (1 mark)

This is the required differential equation.

3. Evaluate : $\int \frac{dx}{\sqrt{x^2+6x+5}}$

Ans. Let I $= \int \frac{dx}{\sqrt{x^2+6x+5}}$

$$= \int \frac{dx}{\sqrt{x^2+6x+9-4}}$$

$$= \int \frac{dx}{\sqrt{(x+3)^2-(2)^2}}$$ (1 mark)

$$\therefore I = \log\left|(x+3)+\sqrt{x^2+6x+5}\right| + C$$ (1 mark)

4 Given X ~ B(n , p) . If n = 25 , E(X) = 10 . Find p and S.D (X) .

Ans. $n = 25, \text{E(X)=10=np}, \quad p = \frac{10}{25} \therefore p = 0.4$ (1 mark)

and $q = 1 - p = 0.6$

$S.D.(X) = +\sqrt{npq} = \sqrt{25 \times 0.4 \times 0.6} = 2.4494$ (1 mark)

5. The p.d.f of a random variable X is given by f(x) = 2x ; $0 \le x \le 1$

= 0 ; otherwise

Then find P (1/3 < X < ½)

Ans. $\text{p}\left(\frac{1}{3} < X < \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x)dx = \int_{1/3}^{1/2} 2x \; dx$ (1 mark)

$$= \left[x^2\right]_{1/3}^{1/2}$$

$$= \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$$ (1 mark)

Q. 5 (A) Attempt any Two : **(6 marks)**

1. Solve the different equation $(x+y)\frac{dy}{dx} = y$.

Ans. $(x+y)\ \frac{dy}{dx} = y$

Put $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\therefore (x+ux)\left(u + x.\frac{du}{dx}\right) = ux$$ (1 mark)

$$\therefore u + u^2 + x.(1+u)\ \frac{du}{dx} = u$$

$$\frac{1+u}{u^2}du + \frac{dx}{x} = 0$$

$$\int\frac{du}{u^2} + \int\frac{du}{u} + \int\frac{dx}{x} = C$$ (1 mark)

$$\frac{-1}{u} + \log u + \log x = C$$

$$\frac{-1}{\frac{y}{x}} + \log\left(\frac{y}{x}\right) + \log x = C$$

$\log y - \frac{x}{y} = C$ is the general solution. (1 mark)

2. Find $\frac{dy}{dx}$, if $y=(\tan x)^x + (4)^{\sin x}$

Ans. $y = (\tan x)^x + (4)^{\sin x}$

$y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots\ldots(i)$$

$u = (\tan x)^x$

$\log u = x.\log(\tan x)$

Differentiating both sides w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = x.\frac{\sec^2 x}{\tan x} + \log(\tan x)$$

$$\frac{du}{dx} = (\tan x)^x\left[\frac{x.\sec^2 x}{\tan x} + \log(\tan x)\right]$$ (1 mark)

$v - 4^{\sin x}$

$$\frac{dv}{dx} = \log 4.\ \cos x.\ 4^{\sin x} \qquad \ldots\ldots.(iii)$$ (1 mark)

From (i), (ii) and (iii)

$$\frac{dy}{dx} = (\tan x)^x\left[\frac{x.\sec^2 x}{\tan x} + \log(\tan x)\right] + \log 4.\cos x.4^{\sin x}$$ (1 mark)

3. Prove that $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Ans. L.H.S

Since 0 < a < 2a

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$ (1 mark)

In the second integral on R.H.S.

Put $x=2a-t$ $\quad \therefore dx=-dt$

It $x = a, t=a$ and $x=2a,\ t=0$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)(-dt)$$ (1 mark)

$$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)(dt)$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$$

$$\left(\because \int_0^a f(x)\,dx = -\int_a^0 f(x)\,dx \right)$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\left(\because \int_0^a f(x)\,dx = \int_0^a f(t)\,dt \right)$$ (1 mark)

Hence proved

Q.5 (B) Attempt any Two : **(8 marks)**

1. Discuss the continuity of the function $f(x)$ on its domain.

Where $f(x) = x^2 + 4$, for $0 \le x \le 2$

$= 3x + 2$, for $2 < x < 4$

$= x^2 + 1$, for $4 \le x \le 6$

Ans. Domain of the function $f(x)$ is [0, 6].

Since $f(x)$ is a polynomial function so it is continuous at every point of its domain except possibly at $x = 2$ and $x = 4$. (1 mark)

We have to discuss the continuity at $x = 2$ and $x = 4$.

At $x = 2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 4)$

$= 4 + 4 = 8$ ……(i)

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x + 2)$

$= 6 + 2 = 8$ ……(ii)

$f(2) = (2)^2 + 3 = 4 + 4 = 8$ ……(iii)

From (i), (ii) and (iii)

$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$

$\therefore$ f is continuous at $x = 2$ (1 mark)

At $x = 4$,

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (3x + 2)$

$= 12 + 2 = 14$ …….(iv)

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (x^2 + 1)$

$= 16 + 1 = 17$ …….(v)

$f(4) = (4)^2 + 1 = 17$ …… (vi)

From (iv), (v) and (vi)

$\lim_{x \to 4^-} f(x) \ne f(4) = \lim_{x \to 4^+} f(x)$

$\therefore f$ is discontinuous at $x = 4$ (1 mark)

Function f is continuous at every point of its domain [0, 6] except at $x = 4$. (1 mark)

2. Find the particular solution of the differential equation. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

Ans. $(e^y + 1)\cos x\;dx + e^y \sin x\;dy = 0$

$$\frac{\cos x}{\cos x}\,dx + \frac{e^y}{e^y+1}\,dy = 0$$ (1 mark)

$$\int \frac{\cos x}{\sin x}\,dx + \int \frac{e^y}{e^y+1}\,dy = \log c$$

$$\log(\sin x) + \log(e^y+1) = \log c$$

$$\log\left[\sin x.(e^y+1)\right] = \log c$$

$$\sin x.(e^y+1) = c$$ (1 mark)

When $x = \frac{\pi}{4},\ y = 0$ we get

$$c = \sqrt{2}$$ (1 mark)

$\therefore$ The particular solution is

$$\sin x.(e^y+1) = \sqrt{2}$$ (1 mark)

3. Evaluate : $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+a)}}$

Ans. Let I = $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+a)}}$

$$= \int \frac{dx}{\sqrt{\sin^3 x.(\sin x \cos a + \cos x \sin a)}}$$

$$= \int \frac{dx}{\sqrt{\sin^4 x(\cos a + \cot x \sin a)}}$$ (1 mark)

$$= \int \frac{\cos ec^2 x\,dx}{\sqrt{\cos a + \cot x\ \sin a}}$$ (1 mark)

Put $\cos a + \cot x \sin a = t$

$$-\sin a\ \cos ec^2 x\ dx = dt$$

$$\cos ec^2 x\ dx = \frac{-dt}{\sin a}$$

$$\therefore \quad I = \int \frac{\left(\frac{-dt}{\sin a}\right)}{\sqrt{t}}$$

$$= \frac{-1}{\sin a} \int \frac{dt}{\sqrt{t}}$$ (1 mark)

$$= \frac{-2}{\sin a}\sqrt{t} + C$$

$$\therefore I = \frac{-2}{\sin a}\sqrt{\cos a + \cot x \sin a} + C$$ (1 mark)

Q. 6 (A) Attempt any Two : **(6 marks)**

1. If x and y are differentiable functions of t so that y is a function of x, then prove that

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)},\ \text{Where } \frac{dx}{dt} \neq 0$$

Ans. Let δx and δy be the increment in x and y respectively corresponding to the increment δt in t.

Now x and y are differentiable functions of t.

$\therefore \frac{dx}{dt} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t}$ and $\frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$ (1 mark)

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\lim_{\delta t \to 0} \frac{\delta y}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta x}{\delta t}}$$

$$= \lim_{\delta t \to 0} \frac{\delta y}{\delta x}$$

Also as $\delta t \to 0$ we have $\delta x \to 0$

$\therefore \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ exists (1 mark)

$$= \frac{dy}{dx}$$

$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, where $\frac{dx}{dt} \neq 0$ (1 mark)

Hence proved

2. If u and v are function of x, then prove that $\int u.v.dx = u\int vdx - \int\left[\int vdx\right]\frac{du}{dx}dx$

Ans. Let $\int vdx = \omega$

$$\therefore v = \frac{d\omega}{dx}$$

$$\frac{d}{dx}(u.\omega) = u.\frac{d\omega}{dx} + \omega.\frac{du}{dx}$$

$\frac{d}{dx}(u.\omega) = u.v + \omega.\frac{du}{dx}$ (1 mark)

By definition of integral

$$u.\omega = \int\left[u.v + \omega.\frac{du}{dx}\right].dx$$

$$u.\omega. = \int u.vdx + \int \omega.\frac{du}{dx}.dx$$

$\int u.v.dx = u.\omega - \int \omega.\frac{du}{dx}.dx$ (1 mark)

But $\omega = \int v\ dx$

$\therefore \int u.v.dx = u\int vdx - \int\left[\int vdx\right]\frac{du}{dx}\ dx$ (1 mark)

Hence proved

3. The probability of hitting a target in any shot is 0. 2 . If 10 shots are fired,

find the probability that the target will be hit at least twice

Ans. Let X = number of shots hitting the target

$p =$ probability that the target is shot

$\therefore p = 0.2 = \frac{1}{5}$

$\therefore q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

Given : $n = 10$

$\therefore X \sim B\left(10, \frac{1}{5}\right)$

The p.m.f. of X is given as (1 mark)

$p(X = x) = {}^{n}C_{x} p^{x} q^{n-x}$

i.e., $p(x) = {}^{10}C_{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{10-x}, \qquad x = 0,1,2,.....,10$

p (target will be hit at least twice) (1 mark)

$= p(X \geq 2) = 1 - \left\{p\left[X = 0\right] + p\left[X = 1\right]\right\}$

$= 1 - \left[p(0) + p(1)\right]$

$= 1 - \left[{}^{10}C_{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{10-0} + {}^{10}C_{1}\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{10-1}\right]$

$= 1 - \left[(1)\left(\frac{4}{5}\right)^{10} + 10\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{9}\right]$

$= 1 - \left[\frac{4}{5} + 2\right]\left(\frac{4}{5}\right)^{9} = 1 - 2.8 \times 0134217728$

$= 1 - 0.3758096384 = 0.6241903616$

$\therefore p\left[X \geq 2\right] = 0.6241$

Hence, the probability that the target will be hit at least twice is 0.6241. (1 mark)

Q.6 (B) Attempt any Two : **(8 marks)**

1. Show that $\int_{1}^{3} \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}}\, dx = 1$

Ans. Let I = $\int_{1}^{3} \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}}\, dx$.....(i)

By using property

$$\int_{a}^{b} f(x)\, dx = \int_{a}^{b} f(a+b-x)dx$$ (1 mark)

$$I = \int_{1}^{3} \frac{\sqrt[3]{(1+3)-x+5}}{\sqrt[3]{(1+3)-x+5} + \sqrt[3]{9-(1+3-x)}}\, dx$$

$$= \int_{1}^{3} \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x} + \sqrt[3]{x+5}}\, dx \quad ....(ii)$$ (1 mark)

Adding (i) and (ii),

$$2I = \int_1^3 \left(\frac{\sqrt[3]{x+5} + \sqrt[3]{9-x}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}} \right) dx$$

$$= \int_1^3 1. \; dx \qquad \text{(1 mark)}$$

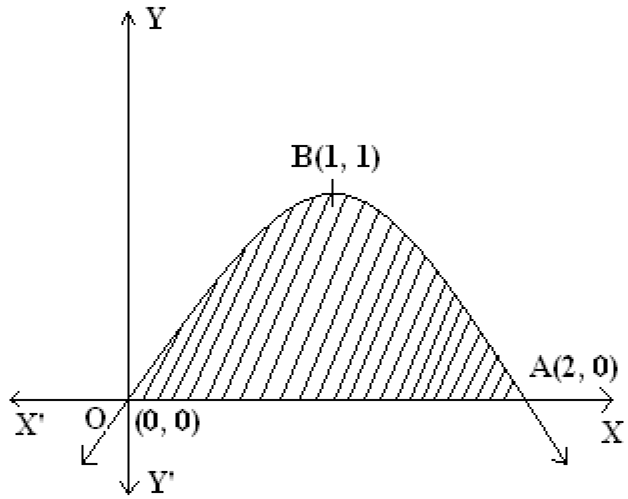
$$= [x]_1^3$$

$\therefore 2I = 2 \qquad \therefore I = 1$ (1 mark)

$$\therefore \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}} \, dx = 1$$

2. Find the area of the region bounded by the curve $y = 2x - x^2$ and X-axis.

Ans.



The equation of the curve (i.e., parabola) can be written as :

$x^2 - 2x = -y$ (1 mark)

$\therefore x^2 - 2x + 1 = -y + 1$

$\therefore (x-1)^2 = -(y-1)$

$\therefore$ The parabola has its vertex at B(1, 1) and it is opening downwards.

It intersects the X-axis, where $y = 0$.

$\therefore$ putting $y = 0$ in the equation $y = 2x - x^2$, we get (1 mark)

$0 = 2x - x^2 = x(2-x)$

$\therefore x = 0, \quad x = 2$

$\therefore$ the points of intersection of the parabola with the X-axis are O(0, 0) and A(2, 0).

$\therefore$ required area = area of the region OABO

$$= \int_0^2 y \, dx = \int_0^2 (2x - x^2) \, dx \qquad \text{(1 mark)}$$

$$= \left[2.\frac{x^2}{2} - \frac{x^3}{3} \right]_0^2 = \left[x^2 - \frac{x^3}{3} \right]_0^2$$

$$= \left(4 - \frac{8}{3} \right) - 0 = \frac{4}{3} \text{ sq units.} \qquad \text{(1 mark)}$$

3. Let X = time (in minute) that lapses between the bell and the end of the lectures in case of a college professor. Suppose X has p.d.f.

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of k.

(ii) What is the probability that lecture ends within 1 minute of the bell ringing?

(iii) What is the probability that lecture continues for at least 90 seconds beyond the bell?

Ans. (i) Since $f(x)$ is p.d.f.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$ (1 mark)

$$\therefore 0 + \int_{0}^{2} f(x)\,dx + 0 = 1 \quad \ldots\ldots..[\because\ f(x) = 0,\ \text{when}\ x < 0\ \text{and}\ x > 2]$$

$$\therefore \int_{0}^{2} kx^2\,dx = 1$$

$$\therefore k\int_{0}^{2} x^2\,dx = 1$$

$$\therefore k\left[\frac{x^3}{3}\right]_0^2 = 1 \qquad \therefore \frac{k}{3}[x^3]_0^2 = 1$$

$$\therefore \frac{k}{3}(8-0) = 1 \qquad \therefore \frac{8}{3}k = 1$$

$$\therefore k = \frac{3}{8}.$$ (1 mark)

(ii) Requires probability = $P(X \le 1) = \int_{-\infty}^{1} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= 0 + \int_{0}^{1} f(x)\,dx \quad \ldots.\ [\because f(x) = 0,\ \text{when}\ x < 0]$$

$$= \int_{0}^{1} kx^2\,dx = k\int_{0}^{1} x^2\,dx$$

$$= k\left[\frac{x^3}{3}\right]_0^1 = \frac{k}{3}\left[x^3\right]_0^1$$

$$= \frac{k}{3}\left[1-0\right] = \frac{k}{3} = \frac{1}{3}\times\frac{3}{8} = \frac{1}{8}.$$ (1 mark)

(iii) Required probability = $P(X \ge 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$$= \int_{1.5}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= \int_{1.5}^{2} f(x)\, dx + 0 \qquad \ldots [\because f(x) = 0, \text{when } x > 2]$$

$$= \int_{1.5}^{2} kx^2\, dx = k\int_{1.5}^{2} x^2\, dx$$

$$= k\left[\frac{x^3}{3}\right]_{1.5}^{2} = \frac{k}{3}\left[x^3\right]_{1.5}^{2}$$

$$= \frac{k}{3}[8 - 3.375] = \frac{1}{3} \times \frac{3}{8} \times 4.625$$ (1 mark)

$$= 0.578$$