MATHS PAPER II (SOLUTION)

39. (C)

[Sol. $\int \frac{\cos x - x\sin x}{\sqrt{x\cos x}}dx$; put $x\cos x = t \Rightarrow (\cos x - x\sin x)\,dx = 2t\,dt$

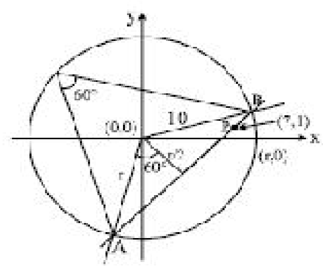
$= \int \frac{2t\,dt}{t} = 2t + C = 2\sqrt{x\cos x} + C$ **Ans.**]

40. (A)

[Sol. Equation of line : $y - 1 = m(x - 7)$

$mx - y + 1 - 7m = 0$

Perpendicular distancer from (0, 0)= $\frac{r}{2} \Rightarrow \frac{|7m-1|}{\sqrt{1+m^2}} = \frac{r}{2} = 5$



$(7m - 1)^2 = 25(1 + m^2)$

$49m^2 - 14m + 1 = 25 + 25m^2$

$24m^2 - 14m - 24 = 0 \Rightarrow m_1m_2 = -1$ **Ans.**]

41. (B)

[Sol. Clearly $f(0) = 1 > 0$

& $f(1) = 2 - n < 0$ $(\because n \geq 3)$

$\therefore$ $f(x) = 0$ will have at least one root in (0, 1).

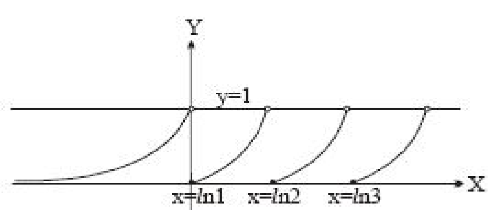
Statement-2 is also correct but not correct explanation to Statement-1 $\Rightarrow$ **(B)**]

42. (D)

[Sol. $y = \{e^x\} = \begin{cases} e^x & ; \quad -\infty < x < 0 \\ e^x - 1 & ; \quad 0 \leq x < \ln 2 \\ e^x - 2 & ; \quad \ln 2 \leq x < \ln 3 \end{cases}$

and so on

Clearly f(x) is aperiodic on R.]



Graph of $y = \{e^x\}$

43. (B,C,D)

[Sol. We have $f'(x) = 1 + \cos x \Rightarrow$ f is strictly increasing and has inflection point at $x = (2n + 1)\pi$

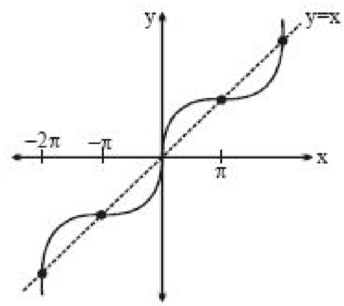
Also there is no x for which f (x) is not increasing.

$\Rightarrow$ (A) is not correct

As f is continuous $\forall\, x \in R$ hence bounded in every closed interval.

As f is odd hence symmetric w.r.t origin and its graphs lies in 1st and 3rd quadrant and y = x cut the graph at infinitely many points.

$\Rightarrow$ **B, C, D** are correct]



44. (A,B,C,D)

[Sol. We have $f(2 - x) = f(2 + x)$

Replacing x by $2 - x$, we get

$$f(x) = f(4 - x) \quad ....(1)$$

Put $x = -4$ in (1), we get

$$f(-4) = f(8) \quad \Rightarrow \quad \textbf{(A) is correct}$$

On differentiating (1) w.r.t. x, we get

$$f'(x) = -f'(4 - x) \quad ....(2)$$

Put $x = \frac{1}{2}, 1, 2$ in (2), we get

$$f'\left(\frac{1}{2}\right) = 0 = f'(1) = f'(2) = f'\left(\frac{7}{2}\right) = f'(3)$$

Now, consider a function $y = f'(x)$

As $f'(x)$ satisfy Rolle's theorem in $\left[\frac{1}{2},1\right],[1,2],\left[2,\frac{7}{2}\right],\left[\frac{7}{2},3\right]$ respectively.

So, by Rolle's theorem, the equation $f''(x)=0$ has minimum 4 roots in $(0, 4)$. $\Rightarrow$ **(B) is correct**

Now, consider $I_1=\int\limits_{-\pi/4}^{\pi/4} f(2+x)\sin x\,dx$ ....(3)

$$I_1=\int\limits_{-\pi/4}^{\pi/4} f(2-x)\ \sin(-x)\,dx=-\int\limits_{-\pi/4}^{\pi/4} f(2+x)\ \sin(x)\,dx$$

$\therefore\quad I_1=-I_1$

Hence $I_1=0$ $\Rightarrow$ **(C) is correct**

Again, consider $I_2=\int\limits_0^2 f(t)\ 5^{\cos\pi t}\,dt$

Put $4-t=y \Rightarrow dt=-dy$

So, $I_2=\int\limits_4^2 f(4-y)\ 5^{\cos\pi(4-y)}\,(-dy)=\int\limits_2^4 f(4-y)\ 5^{\cos\pi y}\,dy=\int\limits_2^4 f(4-t)\ 5^{\cos\pi t}\,dt$ $\Rightarrow$ **(D) is correct]**

45. (B.C)

[Sol. $P(k)=\left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\left(\frac{\pi}{2}-\frac{\pi}{4k}\right)\right)\left(1+\cos\left(\frac{\pi}{2}+\frac{\pi}{4k}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{4k}\right)\right)$

$$=\left(1+\cos\frac{\pi}{4k}\right)\left(1+\sin\frac{\pi}{4k}\right)\left(1-\sin\frac{\pi}{4k}\right)\left(1-\cos\frac{\pi}{4k}\right)$$

$$=\left(1-\cos^2\frac{\pi}{4k}\right)\left(1-\sin^2\frac{\pi}{4k}\right)=\frac{4\sin^2\frac{\pi}{4k}\cdot\cos^2\frac{\pi}{4k}}{4}$$

$P(k)=\frac{1}{4}\sin^2\left(\frac{\pi}{2k}\right) \Rightarrow P(3)=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16};$

$P(4)=\frac{\pi}{4}\sin^2\frac{\pi}{2k}=\frac{1}{4}\sin^2\frac{\pi}{8}=\frac{1}{8}\left(1-\cos\frac{\pi}{4}\right)=\frac{2-\sqrt{2}}{16} \Rightarrow$ **(B)**

$P(5)=\frac{1}{4}\sin^2\frac{\pi}{10}=\frac{1}{8}\left(2\sin^2\frac{\pi}{10}\right)=\frac{1}{8}(1-\cos 36^\circ)=\frac{1}{8}\left(1-\frac{\sqrt{5}+1}{4}\right)=\frac{3-\sqrt{5}}{32} \Rightarrow$ **(C)**

$P(6)=\frac{1}{4}\sin^2\frac{\pi}{12}=\frac{1}{8}\left(2\sin^2\frac{\pi}{12}\right)=\frac{1}{8}\left(1-\cos\frac{\pi}{6}\right)=\frac{1}{8}\left(1-\frac{\sqrt{3}}{2}\right)=\frac{2-\sqrt{3}}{16}$]

46. (C,D)

[Sol. for continuity at $x=0$

$f(0)=0$; $f(0^-)=0$; $f(0^+)=\lim\limits_{h\to 0} h^n\sin\frac{1}{h}=0 \Rightarrow n>0$

for derivability at $x=0$

$f'(0^-)=0$; $f'(0^+)=\lim\limits_{h\to 0}\frac{h^n\sin\frac{1}{h}}{h}$ for this limit not to exist $n\le 1$

hence $0<n\le 1$

47. ***(A,D)***

[Sol. $g'(x) = ae^{ax} + f'(x)$

$g'(0) = a + f'(0) = a - 5$ ....(1)

$g''(x) = a^2e^{ax} + f''(x)$

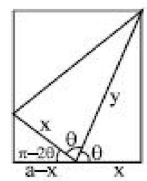
$g''(0) = a^2 + f''(0) = a^2 + 3$ ....(2)

$g'(0) + g''(0) = 0$

$\Rightarrow \quad a - 5 + a^2 + 3 = 0 \quad \Rightarrow \quad a^2 + a - 2 = 0 \quad \Rightarrow \quad (a+2)(a-1) = 0 \quad \Rightarrow \quad a = 1, -2$ **Ans.]**

48. (A)-R , (B)-Q , (C)-P,Q,R,S,T , (D)- R,S,T

[Sol.**(A)** $\cos\theta = \dfrac{x}{y}$



$-\cos 2\theta = \dfrac{a-x}{x}$

$1 - 2\dfrac{x^2}{y^2} = \dfrac{a-x}{x}; \quad \dfrac{2x^2}{y^2} = 1 - \dfrac{a-x}{x}; \quad \dfrac{2x^2}{y^2} = \dfrac{2x-a}{x}$

$f(x) = y^2 = \dfrac{2x^3}{2x-a}; \quad f'(x) = 2\left[\dfrac{(2x-a)3x^2 - x^3\cdot 2}{(2x-a)^2}\right]$

$3(3x-a) = 2x; \quad 4x = 3a$

$$y^2 = \frac{2\cdot 27a^3}{64\left(\frac{3a}{2} - a\right)} = \frac{2\cdot 27}{64}\cdot\frac{2}{a}a^3 = \frac{27}{16}\cdot\frac{16}{3} = 9 \qquad \Rightarrow \qquad y = 3 \textbf{ Ans.}$$

Alternatively: EF = x

BC + BE + EC

$\dfrac{4}{\sqrt{3}} = -x\cos\theta\cos 2\theta + x\cos\theta; \qquad x = \dfrac{4/\sqrt{3}}{\cos\theta(1 - \cos 2\theta)}$

after differentiating it maximum of denominator at $\theta = \sin^{-1}\sqrt{\dfrac{2}{3}}$, so required $x = \dfrac{3\sqrt{3}(4/\sqrt{3})}{4} = 3$ Ans.

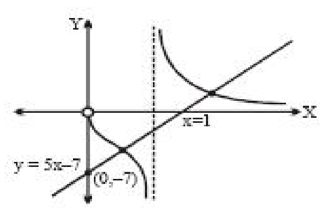
(B) Let $f(x) = \log_x 3 = \dfrac{\ln 3}{\ln x}$

Domain of $f(x) = (0,1) \cup (1,\infty)$

$\Rightarrow \quad f'(x) = \dfrac{-1}{(\ln x)^2} \times \ln 3 \times \dfrac{1}{x} < 0$

$\therefore$ f(x) is decreasing function in $(0,1) \cup (1,\infty)$

From graph, clearly the above equation has two solutions.



(C) Clearly $f(x) > 0 \ \forall x \in (0, \infty)$

Now $f(x) = \dfrac{96x}{\sqrt{9x^2+173x+900}+\sqrt{9x^2+77x+900}}$ (Rationalise)

$\therefore \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{96}{\sqrt{9+\frac{173}{x}+\frac{900}{x^2}}+\sqrt{9+\frac{77}{x}+\frac{900}{x^2}}} = 16$

$\therefore \quad R_f = (0, 16)$ **Ans.** $\Rightarrow$ **P, Q, R, S, T**

(D) Consider 3 number x^2, y^2 and z^2 and using

R.M.S $\ge$ A.M.

$$\sqrt{\frac{x^4+y^4+z^4}{3}} \ge \frac{(x^2+y^2+z^2)}{3}$$

$\therefore \quad (x^2+y^2+z^2)^2 \le 3(x^4+y^4+z^4)$

hence $k \ge 3$, equality holds if $x^2 = y^2 = z^2$ $\Rightarrow$ **R, S, T**

49. (A)-T,(B)-Q,(C)-Q,R,S,T,(D)-R,S

[Sol.

(A) $\dfrac{e^{\ln x!}}{(x-2)!} = 20 \Rightarrow \dfrac{x!}{(x-2)!} = 20 \Rightarrow x(x-1) = 20 \Rightarrow x^2 - x - 20 = 0 \Rightarrow (x-5)(x+4) = 0$

$\therefore \quad x = 5$ **Ans.** $\Rightarrow$ **(T)**

(B) $\because \quad 4\{x\} = x + [x]$ ....(1)

$$= x + x - \{x\} \Rightarrow 5\{x\} = 2x \Rightarrow \{x\} = \frac{2x}{5} \Rightarrow 0 \le \frac{2x}{5} < 1 \Rightarrow 0 \le x < \frac{5}{2}$$

hence $[x] = 0, 1, 2$

Again from (1) $4x - 4[x] = x + [x]$

$3x = 5[x]$

Case-I: If $x \in [0, 1) \Rightarrow [x] = 0$

$\therefore \quad 3x = 0 \Rightarrow \boxed{x = 0}$

Case-II: If $x \in [1, 2) \Rightarrow [x] = 1$

$\therefore \quad 3x = 5 \Rightarrow \boxed{x = \frac{5}{3}}$

Case-III: If $x \in (2, 5/2] \Rightarrow [x] = 2$

$\therefore \quad 3x = 10 \Rightarrow x = \frac{10}{3}$ (reject)

$\therefore$ number of solutions = 2 **Ans.**

Alternatively :

Let x = I + f

$\Rightarrow \quad 4f = I + f + I \quad \Rightarrow \quad f = \frac{2I}{3} \quad \Rightarrow \quad I = 0, 1 \quad \Rightarrow \quad x = 0 \ \& \ x = 1 + \frac{2}{3} = \frac{4}{3}$

(C) Given equation is

$x^2 + 2x(y + g) + y^2 + 2fy + 4 = 0$

$2x = -\ 2(y + g) \pm \sqrt{4(y+g)^2 - 4(y^2 + 2fy + 4)}$

$x = -(y + g) \pm \sqrt{(g^2 - 4) + 2y(g - f)}$ ....(1)

(1) will represent a pair of lines if its discriminant is zero

$\Rightarrow \quad +4(g - f) = 0 \quad \Rightarrow \quad g = f$

$\therefore \quad x = -(y + g) \pm \sqrt{g^2 - 4}$. For two real lines

$g^2 \ge 4 \quad \Rightarrow \quad g \ge 2$ or $g \le -2$ **Ans.** $\Rightarrow$ **Q, R, S, T**

(D) If d = 6; A = (0, 0); B(6, 0)

Consider a circle with centre A are radius $2(r_1)$ and a circle with centre B and radius $3(r_2)$. The circles will be separated. There will 4 common tangents at a distance of 2 from A and 3 from B $\Rightarrow$ 4 lines

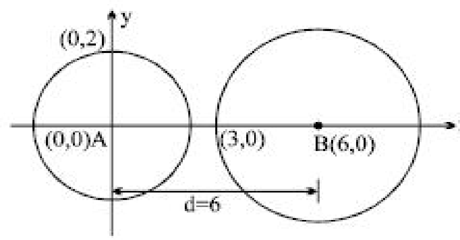
|||ly if d = 5; $r_1 + r_2 = 5 \Rightarrow$ circles touches externally

$\Rightarrow$ 3 common tangents $\Rightarrow$ 3 lines $\Rightarrow$ **R, S**

NOTE: |||ly for two intersecting circles

$r_2 - r_1 < d < r_1 + r_2$ i.e. $1 < d < 5 \Rightarrow$ 2 common tangents $\Rightarrow$ 2 lines

if $d = r_2 - r_1$ i.e. if $d = 1 \Rightarrow$ circles touches internally $\Rightarrow$ 1 lines]



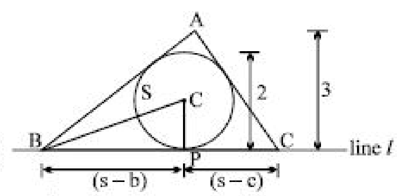
50. (3)

[Sol. $(PB)(PC) = (s - b)(s - c) = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$

$= \frac{\Delta \cdot \Delta}{s(s-a)} = r \cdot \frac{\Delta}{(s-a)} \quad (r = 1)$

$= \frac{\Delta}{(s-a)} = \frac{\Delta}{\Delta - a}$

$= \frac{3a}{2\left(\frac{3a}{2} - a\right)} = \frac{3}{3-2} = 3$ Ans.]



$\left(r = \frac{\Delta}{s} = 1 \Rightarrow s = \Delta\right)$

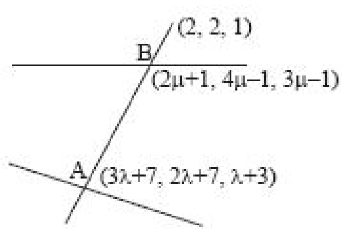
51. (18)

[Sol. $\frac{3\lambda - 2\mu + 6}{2} = \frac{2\lambda - 4\mu + 8}{2} = \frac{\lambda - 3\mu + 4}{1}$

On solving we get $\mu = 0$ and $\lambda = 2$

Hence A(13, 11, 5); B(1, – 1, – 1)

$AB = \sqrt{144 + 144 + 36} = \sqrt{324} = 18$ **Ans.**]



52. (2187)

[Sol. We know that $(a-1)^n = {}^nC_0\, a^n - {}^nC_1\, a^{n-1} + {}^nC_2\, a^{n-2} - \ldots\ldots\ldots\ldots + (-1)^{n-1}\, {}^nC_{n-1} a + (-1)^n\, {}^nC_n$

$$\therefore \quad \frac{(a-1)^n}{a} = {}^nC_0 a^{n-1} - {}^nC_1\, a^{n-2} + {}^nC_2\, a^{n-3} - \ldots\ldots\ldots\ldots + (-1)^{n-1}\, {}^nC_{n-1} + \frac{(-1)^n}{a}\, {}^nC_n$$

Hence $f(n) = \dfrac{(a-1)^n - (-1)^n}{a}$

Now, $f(2007) + f(2008) = \dfrac{(a-1)^{2007}+1}{a} + \dfrac{(a-1)^{2008}-1}{a} = \dfrac{(a-1)^{2007}(1+a-1)}{a} = (a-1)^{2007}$

$$= \left(3^{\frac{1}{223}}\right)^{2007} = 3^9 = 3^2 \cdot 3^7 = 9(2187)$$

Hence $k = 2187$ **Ans.**

Alternatively: $f(n) = \dfrac{1}{a}\left[{}^nC_0 \cdot a^n - {}^nC_1 \cdot a^{n-1} + {}^nC_2 \cdot a^{n-2} - \ldots\ldots + (-1)^{n-1} \cdot {}^nC_{n-1} \cdot a + (-1)^n\right] - \dfrac{(-1)^n}{a}$

$$\therefore \quad f(n) = \frac{1}{a}[a-1]^n - \frac{(-1)^n}{a} \qquad \ldots(1)$$

now given $a = 3^{\frac{1}{223}} + 1$ or $a - 1 = 3^{\frac{1}{223}}$

hence $f(n) = \dfrac{1}{a}\left(3^{\frac{1}{223}}\right)^n - \dfrac{(-1)^n}{a} \qquad \ldots(2)$

$$\therefore \quad f(2007) = \frac{1}{a}\left(3^{\frac{1}{223}}\right)^{2007} - \frac{(-1)^{2007}}{a}$$

or $\quad f(2007) = \dfrac{1}{a}(3^9) + \dfrac{1}{a} \qquad \left(\dfrac{2007}{223} = 9\right)$

$$f(2007) = \frac{1}{a}(3^9 + 1) \quad \ldots(3)$$

$$f(2008) = \frac{1}{a}\left(3^{\frac{2008}{223}}\right) - \frac{1}{a} = \frac{1}{a}\left(3^9 \cdot 3^{\frac{1}{223}}\right) - \frac{1}{a} \text{ or } f(2008) = \frac{1}{a}\left(3^9 \cdot 3^{\frac{1}{223}} - 1\right) \quad \ldots(4)$$

hence $f(2007) + f(2008) = \dfrac{1}{a}(3^9+1) + \dfrac{1}{a}\left(3^9 \cdot 3^{\frac{1}{223}} - 1\right) = \dfrac{1}{a}\left(3^9 + 3^9 \cdot 3^{\frac{1}{223}}\right)$

$$= \frac{3^9}{a}\left(1 + 3^{\frac{1}{223}}\right) = \frac{3^9}{a} \cdot a = 3^9 = 9 \cdot 3^7 = 9(2187) \Rightarrow k = 2187 \textbf{ Ans.}$$

53. (7)

[Sol. Let $I = \int_{-1}^{1} \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)\left(\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right)dx$ ......(1)

$$I = \int_{-1}^{1}\left(\cot^{-1}\frac{1}{\sqrt{1-x^2}}\right)\left(\cot^{-1}\frac{-x}{\sqrt{1-(x^2)^{|x|}}}\right)dx \quad ....(2)$$

On adding

$$2I = \int_{-1}^{1}\cot^{-1}\frac{1}{\sqrt{1-x^2}}\left\{\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}+\pi-\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right\}dx$$

$$2I = \int_{-1}^{1}\pi\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)dx = \int_{-1}^{1}\pi\tan^{-1}\sqrt{1-x^2}\,dx = 2\pi\int_{0}^{1}\tan^{-1}\sqrt{1-x^2}\,dx \quad ....(3)$$

$\left(\text{As } \tan^{-1}\sqrt{1-x^2} \text{ is even function}\right)$

$$\therefore \quad I = \pi\int_{0}^{1}\underbrace{1}_{II}\underbrace{\tan^{-1}\left(\sqrt{1-x^2}\right)}_{I}dx \quad ....(4)$$

Integrating by parts

$$I = \pi\underbrace{\tan^{-1}\left(\sqrt{1-x^2}\right)\cdot x\Big]_0^1}_{zero} - \int_0^1\frac{x}{(1+1-x^2)}\frac{(-x)}{\sqrt{1-x^2}}dx = \pi\int_0^1\frac{x^2}{(2-x^2)\sqrt{1-x^2}}dx$$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$I = \pi\int_0^{\pi/2}\frac{\sin^2\theta}{(2-\sin^2\theta)}d\theta = -\pi\int_0^{\pi/2}\frac{2-\sin^2\theta-2}{2-\sin^2\theta}d\theta$$

$$\therefore \quad I = 2\pi\int_0^{\pi/2}\frac{d\theta}{2-\sin^2\theta}-\frac{\pi^2}{2}$$

$$I = 2\pi\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{2+2\tan^2\theta-\tan^2\theta}-\frac{\pi^2}{2} = 2\pi\int_0^{\pi/2}\frac{\sec^2\theta\, d\theta}{2+\tan^2\theta}-\frac{\pi^2}{2}$$

put $\tan\theta = t$

$$I = 2\pi\int_0^{\infty}\frac{dt}{2+t^2}-\frac{\pi^2}{2} = 2\pi\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\Big]_0^{\infty}-\frac{\pi^2}{2} = \frac{\pi^2}{\sqrt{2}}-\frac{\pi^2}{2} = \frac{\pi^2\left(\sqrt{2}-1\right)}{2} = \frac{\pi^2\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{c}}$$

$\Rightarrow$ a = 2, b = 1 and c = 4 $\Rightarrow$ a + b + c = 2 + 1 + 4 = 7 **Ans.**]

54. (24)

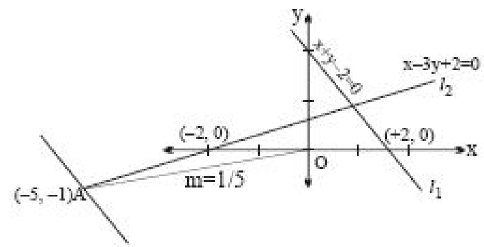
[Sol. $x^2 - 3y^2 - 2xy + 8y - 4 \equiv (x - 3y + 2)(x + y - 2)$
note that $(-5, -1)$ lies on $x - 3y + 2 = 0$
In limiting case line passing through $(-5, -1)$
can be $\parallel$ to $x + y - 2 = 0$
i.e. $m > -1$

and maximum slope can occur if it passes through (0, 0)

i.e. $m < \frac{1}{5} \Rightarrow m \in \left(-1, \frac{1}{5}\right)$

$\Rightarrow a = -1$ and $b = \frac{1}{5}$

Hence $\left(a + \frac{1}{b^2}\right) = -1 + 25 = 24$ **Ans.]**



55. (10)

[Sol. A A A A A | B B B B B B
Middle digit must be A (think !)
so that even number of A's and B's are available
Take AABBB on one side of M (6th place) and then their image about M in a unique way

M
× × × × × ↓ × × × × ×

$\therefore$ Number of ways $= \frac{5!}{2!\cdot 3!} = 10$ **Ans.]**

56. (25)

[Sol. $\frac{1}{5}\cdot y^{\frac{1}{5}-1}\cdot y_1 - \frac{1}{5}\cdot y^{-\frac{1}{5}-1}\cdot y_1 = 2;$ $\qquad \frac{1}{5}\cdot\frac{y^{\frac{1}{5}}\cdot y_1}{y} - \frac{1}{5}\cdot\frac{y^{-\frac{1}{5}}\cdot y_1}{y} = 2$

$y^{\frac{1}{5}} - y^{-\frac{1}{5}} = \frac{10y}{y_1} \quad \Rightarrow \quad \left(y^{\frac{1}{5}} - y^{-\frac{1}{5}}\right)^2 = \frac{100y^2}{y_1^2} \quad \Rightarrow \quad \left(y^{\frac{1}{5}} + y^{-\frac{1}{5}}\right)^2 - 4 = \frac{100y^2}{y_1^2}$

$4(x^2 - 1) = \frac{100y^2}{y_1^2} \quad \Rightarrow \quad y_1^2(x^2 - 1) = 25y^2 \quad \Rightarrow \quad (x^2 - 1)2y_1y_2 + y_1^2\, 2x = 25\cdot 2\cdot y y_1$

$(x^2 - 1)y_2 + xy_1 = 25y$ or $\quad (x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 25y \quad \Rightarrow \quad k = 25$ **Ans.]**

57. (22)